TECHNOLOGICAL AND ECONOMIC DEVELOPMENT OF ECONOMY
ISSN 2029-4913 / eISSN 2029-4921

2016 Volume 22(1): 75–99
doi:10.3846/20294913.2014.984254

CONTINUOUS INTUITIONISTIC FUZZY ORDERED WEIGHTED DISTANCE MEASURE AND ITS APPLICATION TO GROUP DECISION MAKING

Ligang ZHOU[a, b], Feifei JIN[a], Huayou CHEN[a], Jinpei LIU[c]

[a]*School of Mathematical Sciences, Anhui University, Hefei, Anhui 230601, China*
[b]*Signal and Image Processing Institute, Department of Electrical Engineering, University of Southern California, Los Angeles, CA, 90089, USA*
[c]*School of Business, Anhui University, Hefei, Anhui 230601, China*

Received 04 August 2012; accepted 01 June 2013

Abstract. The aim of this paper is to develop the continuous intuitionistic fuzzy ordered weighted distance (C-IFOWD) measure by using the continuous intuitionistic fuzzy ordered weighted averaging (C-IFOWA) operator in the interval distance. We investigate some desirable properties and different families of the C-IFOWD measure. We also generalize the C-IFOWD measure. The prominent characteristics of the C-IFOWD measure are that it is not only a generalization of some widely used distance measure, but also it can deal with interval deviations in aggregation on interval-valued intuitionistic fuzzy values (IVIFVs) by using a controlled parameter, which can decrease the uncertainty of argument and improve the accuracy of decision. The desirable characteristics make the C-IFOWD measure suitable to wide range situations, such as decision making, engineering and investment, etc. In the end, we introduce a new approach to group decision making with IVIFVs in human resource management.

Keywords: group decision making, distance measure, OWA operator, C-IFOWD measure.

JEL Classification: C43, D81.

Introduction

A multiple attribute group decision making (MAGDM) problem is to finding a desirable solution from a finite number of feasible alternatives assessed on multiple attributes by decision makers, both quantitative and qualitative (Wei 2010a). The fundamental prerequisite of MAGDM is how to aggregating individual decision makers' preference information on alternatives (Sengupta, Pal 2009). Information aggregation is a process that combines

Corresponding author Huayou Chen
E-mail: *huayouc@126.com*

individual decision makers' preferences into an overall one by using a proper aggregation technique. A very practical technique for information aggregation is the OWA operator (Yager 1988), which can provide a parameterized family of aggregation operators that includes the maximum, the minimum, the average and other. Since its introduction, the OWA operator has been studied in a wide range of applications and extensions (Calvo *et al.* 2002; Chen *et al.* 2012; Li 2011; Liu 2011; Merigó 2008; Merigó, Gil-Lafuente 2009, 2011a; Merigó *et al.* 2012; Su *et al.* 2012; Wei 2010b; Wei, Zhao 2012a, 2012b; Wu, Cao 2013; Xia *et al.* 2012; Xu, Wang 2012; Xu 2004, 2006a, 2007a, 2010a, 2011; Xu, Cai 2010; Xu, Da 2002a; Xu, Xia 2011a; Yager 2003, 2004a, 2004b; Yager, Kacprzyk 1997; Yager *et al.* 2011; Yang, Chen 2012; Zhao *et al.* 2010; Zhou, Chen 2010, 2011, 2012; Zhou *et al.* 2011, 2012a, 2012b, 2012c).

In order to aggregate the interval arguments, Yager (2004b) introduced a continuous ordered weighted averaging (C-OWA) operator, which is an extension of the OWA operator when the given argument is a continuous valued interval rather than an exact argument. Recently, the C-OWA operator has attracted more and more attentions from both decision makers and researchers (Chen, Zhou 2011, 2012; Wu *et al.* 2009, 2010; Yager, Xu 2006; Zhou, Chen 2011).

Another interesting extension of the OWA is the one that uses distance measures in the OWA operator, which is called the ordered weighted distance (OWD) measure (Xu, Chen 2008a). The main advantage of the OWD measure is that it can relieve (or intensify) the influence of unduly large or unduly small deviations on aggregation results by assigning them low (or high) weights. Motivated by Xu and Chen, Yager (2010) provided a variety of ordered weighted averaging norms based on several similarity measures. Xu (2012) developed some fuzzy ordered distance measures including the linguistic ordered weighted distance measure, uncertain ordered weighted distance measure, linguistic hybrid weighted distance measure, and uncertain hybrid weighted distance measure, etc. Zeng and Su (2011) extended the OWD measure to intuitionistic fuzzy environment and proposed the intuitionistic fuzzy ordered weighted distance (IFOWD) operator. Zhou, Chen and Liu (2012b) presented the continuous ordered weighted distance (COWD) measure by using the C-OWA operator in the interval distance. The use of the different distance measures in different aggregation operators has been studied by several authors (Merigó, Casanovas 2010, 2011a, 2011b; Merigó, Gil-Lafuente 2007, 2010; Xu 2010b, 2012; Xu, Chen 2008b; Xu, Xia 2011b, 2011c; Yue 2011; Zeng, Su 2011; Zhang *et al.* 2011; Zhou *et al.* 2012b).

However, due to the increasing complexity of the socio-economic environment and the lack of knowledge or data about the problem domain, in the process of MAGDM, decision makers may provide their preferences over alternatives with uncertain intuitionistic fuzzy variables when using the distance measures in the aggregation operators. For example, in MAGDM problems, each decision maker provides his/her preferences with interval-valued intuitionistic fuzzy variables. Therefore, it is necessary to extend the distance measures to accommodate situation with interval-valued intuitionistic fuzzy information.

For this purpose, we shall develop a new distance measure called the continuous intuitionistic fuzzy ordered weighted distance (C-IFOWD) measure based on the continuous intuitionistic fuzzy ordered weighted averaging (C-IFOWA) operator. We study some properties and different families of the C-IFOWD measure. We further generalize the C-IFOWD measure and obtain the Quasi C-IFOWD measure and the infinitary C-IFOWD measure.

The prominent characteristic of the C-IFOWD measure is that it provides a parameterized family of distance operators. The decision maker is able to consider the MAGDM problem more clearly according to his/her interest in aggregation process. Another advantage of C-IFOWD measure is that it can relieve (or intensify) the influence of unduly large or unduly small deviations on aggregation results by assigning them low (or high) weights. These characteristics make the C-IFOWD measure suitable to deal with the situations where the input arguments are represented with uncertain linguistic information.

We also present an application of new approach to MAGDM in human resource management. We use different types of the C-IFOWD measure, in which decision maker would have different decisions depending on the particular types of parameters used.

The rest of the paper is organized as follows. In Section 1, we briefly describe some preliminaries. Section 2 presents the C-IFOWD measure and studies some properties and families. We also develop some extensions of the C-IFOWD measure. In Section 3, we present a method for multiple attribute group decision making with the C-IFOWD measure and Section 4 provides an illustrative example. In the last Section we end the paper summarizing the main conclusions.

1. Preliminaries

In this section, we briefly review the intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets, the OWA operator, the GOWA operator, the C-OWA operator and the distance measure.

1.1. Intuitionistic fuzzy sets and Interval-valued intuitionistic fuzzy sets

Intuitionistic fuzzy sets (IFS) introduced by Atanassov (1986) is an extension of the classical fuzzy set, which is suitable to deal with vagueness. It is defined as follows:

Definition 1. Let $X=\{x_1,x_2,\cdots,x_n\}$ be fixed. And an IFS A in X is given as:

$$A=\{<x_i,\mu_A(x_i),\nu_A(x_i)>|\, x_i\in X\}, \tag{1}$$

where $\mu_A(x_i)$ and $\nu_A(x_i)$ represent the membership and non-membership degrees of the element x_i to the set A, respectively. The pair $(\mu_A(x_i),\nu_A(x_i))$ is called the intuitionistic fuzzy value (IFV), and each IFV can be simply denoted as $\alpha_i=(\mu_{\alpha_i},\nu_{\alpha_i})$, where $\mu_{\alpha_i}\in[0,1]$, $\nu_{\alpha_i}\in[0,1]$, $\mu_{\alpha_i}+\nu_{\alpha_i}\le 1$. Additionally, $S(\alpha_i)=\mu_{\alpha_i}-\nu_{\alpha_i}$ and $H(\alpha_i)=\mu_{\alpha_i}+\nu_{\alpha_i}$ are called the score and accuracy degrees of α_i, respectively.

For any three IFVs $\alpha=(\mu_\alpha,\nu_\alpha)$, $\alpha_1=(\mu_{\alpha_1},\nu_{\alpha_1})$ and $\alpha_2=(\mu_{\alpha_2},\nu_{\alpha_2})$, the following operational laws are valid (Xu, Yager 2006):

(1) $\alpha_1\oplus\alpha_2=(\mu_{\alpha_1}+\mu_{\alpha_2}-\mu_{\alpha_1}\mu_{\alpha_2},\nu_{\alpha_1}\nu_{\alpha_2})$;

(2) $\lambda\alpha=(1-(1-\mu_\alpha)^\lambda,{\nu_\alpha}^\lambda)$, $\lambda>0$.

To compare any two IFVs $\alpha_1 = (\mu_{\alpha_1}, \nu_{\alpha_1})$ and $\alpha_2 = (\mu_{\alpha_2}, \nu_{\alpha_2})$, Xu and Yager (2006) introduced a simple method as follows:

(1) If $S(\alpha_1) < S(\alpha_2)$, then $\alpha_1 < \alpha_2$;

(2) If $S(\alpha_1) = S(\alpha_2)$, then

(a) If $H(\alpha_1) = H(\alpha_2)$, then $\alpha_1 = \alpha_2$;

(b) If $H(\alpha_1) < H(\alpha_2)$, then $\alpha_1 < \alpha_2$.

In (1989), Atanassov and Gargov generalized the IFS and defined the interval-valued intuitionistic fuzzy sets (IVIFS) as follows:

Definition 2. Let $X = \{x_1, x_2, \cdots, x_n\}$ be fixed. And an IVIFS $\tilde{A}$ in X is given as

$$\tilde{A} = \{< x_i, \tilde{\mu}_{\tilde{A}}(x_i), \tilde{\nu}_{\tilde{A}}(x_i) > | x_i \in X\}, \tag{2}$$

where the membership degree $\tilde{\mu}_{\tilde{A}}(x_i) \subset [0,1]$ and the non-membership degree $\tilde{\nu}_{\tilde{A}}(x_i) \subset [0,1]$ are intervals, which satisfy $\sup \tilde{\mu}_{\tilde{A}}(x_i) + \sup \tilde{\nu}_{\tilde{A}}(x_i) \le 1$ for all $x_i \in X$.

Let $\tilde{\mu}^L_{\tilde{A}}(x_i)$ and $\tilde{\mu}^U_{\tilde{A}}(x_i)$ be the lower and upper boundaries of $\tilde{\mu}_{\tilde{A}}(x_i)$, and $\tilde{\nu}^L_{\tilde{A}}(x_i)$ and $\tilde{\nu}^U_{\tilde{A}}(x_i)$ be the lower and upper boundaries of $\tilde{\nu}_{\tilde{A}}(x_i)$. Then the IVIFS $\tilde{A}$ is equivalent to the following formula:

$$\tilde{A} = \{< x_i, [\tilde{\mu}^L_{\tilde{A}}(x_i), \tilde{\mu}^U_{\tilde{A}}(x_i)], [\tilde{\nu}^L_{\tilde{A}}(x_i), \tilde{\nu}^U_{\tilde{A}}(x_i)] > | x_i \in X\}, \tag{3}$$

where $0 \le \tilde{\mu}^L_{\tilde{A}}(x_i) \le \tilde{\mu}^U_{\tilde{A}}(x_i) \le 1$, $0 \le \tilde{\nu}^L_{\tilde{A}}(x_i) \le \tilde{\nu}^U_{\tilde{A}}(x_i) \le 1$, $\tilde{\mu}^U_{\tilde{A}}(x_i) + \tilde{\nu}^U_{\tilde{A}}(x_i) \le 1$.

Additionally, the pair

$$(\tilde{\mu}_{\tilde{A}}(x_i), \tilde{\nu}_{\tilde{A}}(x_i)) = ([\tilde{\mu}^L_{\tilde{A}}(x_i), \tilde{\mu}^U_{\tilde{A}}(x_i)], [\tilde{\nu}^L_{\tilde{A}}(x_i), \tilde{\nu}^U_{\tilde{A}}(x_i)])$$

is called an interval-valued intuitionistic fuzzy value (IVIFV), and each IVIFV can be simply denoted as $\tilde{\alpha}_i = (\tilde{\mu}_{\tilde{\alpha}_i}, \tilde{\nu}_{\tilde{\alpha}_i}) = ([\tilde{\mu}^L_{\tilde{\alpha}_i}, \tilde{\mu}^U_{\tilde{\alpha}_i}], [\tilde{\nu}^L_{\tilde{\alpha}_i}, \tilde{\nu}^U_{\tilde{\alpha}_i}])$, where $0 \le \tilde{\mu}^L_{\tilde{\alpha}_i} \le \tilde{\mu}^U_{\tilde{\alpha}_i} \le 1$, $0 \le \tilde{\nu}^L_{\tilde{\alpha}_i} \le \tilde{\nu}^U_{\tilde{\alpha}_i} \le 1$ and $\tilde{\mu}^U_{\tilde{\alpha}_i} + \tilde{\nu}^U_{\tilde{\alpha}_i} \le 1$.

For any three IVIFVs $\tilde{\alpha} = (\tilde{\mu}_{\tilde{\alpha}}, \tilde{\nu}_{\tilde{\alpha}}) = ([\tilde{\mu}^L_{\tilde{\alpha}}, \tilde{\mu}^U_{\tilde{\alpha}}], [\tilde{\nu}^L_{\tilde{\alpha}}, \tilde{\nu}^U_{\tilde{\alpha}}])$, $\tilde{\alpha}_1 = (\tilde{\mu}_{\tilde{\alpha}_1}, \tilde{\nu}_{\tilde{\alpha}_1}) = ([\tilde{\mu}^L_{\tilde{\alpha}_1}, \tilde{\mu}^U_{\tilde{\alpha}_1}], [\tilde{\nu}^L_{\tilde{\alpha}_1}, \tilde{\nu}^U_{\tilde{\alpha}_1}])$ and $\tilde{\alpha}_2 = (\tilde{\mu}_{\tilde{\alpha}_2}, \tilde{\nu}_{\tilde{\alpha}_2}) = ([\tilde{\mu}^L_{\tilde{\alpha}_2}, \tilde{\mu}^U_{\tilde{\alpha}_2}], [\tilde{\nu}^L_{\tilde{\alpha}_2}, \tilde{\nu}^U_{\tilde{\alpha}_2}])$, the following operations have been developed by Xu (2007b):

(1) $\tilde{\alpha}_1 \oplus \tilde{\alpha}_2 = ([\tilde{\mu}^L_{\tilde{\alpha}_1} + \tilde{\mu}^L_{\tilde{\alpha}_2} - \tilde{\mu}^L_{\tilde{\alpha}_1}\tilde{\mu}^L_{\tilde{\alpha}_2}, \tilde{\mu}^U_{\tilde{\alpha}_1} + \tilde{\mu}^U_{\tilde{\alpha}_2} - \tilde{\mu}^U_{\tilde{\alpha}_1}\tilde{\mu}^U_{\tilde{\alpha}_2}], [\tilde{\nu}^L_{\tilde{\alpha}_1}\tilde{\nu}^L_{\tilde{\alpha}_2}, \tilde{\nu}^U_{\tilde{\alpha}_1}\tilde{\nu}^U_{\tilde{\alpha}_2}])$;

(2) $\lambda\tilde{\alpha} = ([1 - (1 - \tilde{\mu}^L_{\tilde{\alpha}})^\lambda, 1 - (1 - \tilde{\mu}^U_{\tilde{\alpha}})^\lambda], [(\tilde{\nu}^L_{\tilde{\alpha}})^\lambda, (\tilde{\nu}^U_{\tilde{\alpha}})^\lambda])$, $\lambda > 0$.

Moreover, in (2007b), Xu introduced the score function

$$S(\tilde{\alpha}) = (\tilde{\mu}^L_{\tilde{\alpha}} - \tilde{\nu}^L_{\tilde{\alpha}} + \tilde{\mu}^U_{\tilde{\alpha}} - \tilde{\nu}^U_{\tilde{\alpha}})/2, \tag{4}$$

to get the score of $\tilde{\alpha}$, and defined the accuracy function

$$H(\tilde{\alpha}) = (\tilde{\mu}^L_{\tilde{\alpha}} + \tilde{\mu}^U_{\tilde{\alpha}} + \tilde{\nu}^L_{\tilde{\alpha}} + \tilde{\nu}^U_{\tilde{\alpha}})/2, \tag{5}$$

to evaluate the accuracy degree of $\tilde{\alpha}$. To compare any two IVIFVs $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$, Xu (2007b) presented a simple method as follows:

(1) If $S(\tilde{\alpha}_1) < S(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 < \tilde{\alpha}_2$;

(2) If $S(\tilde{\alpha}_1) = S(\tilde{\alpha}_2)$, then

(a) If $H(\tilde{\alpha}_1) = H(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 = \tilde{\alpha}_2$;

(b) If $H(\tilde{\alpha}_1) < H(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 < \tilde{\alpha}_2$.

For convenience, throughout this paper, let Ω be the set of all intuitionistic fuzzy values and Σ be the set of all interval-valued intuitionistic fuzzy values.

1.2. The OWA operator and the GOWA operator

The OWA operator (Yager 1988) is an aggregation operator that provides a parameterized family of aggregation operators between the minimum and the maximum. It is defined as follows:

Definition 3. An OWA operator of dimension n is mapping $OWA: R^n \to R$ that has an associated weighting vector $\mathbf{w}$ with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$, such that:

$$OWA(a_1, a_2, \cdots, a_n) = \sum_{j=1}^{n} w_j b_j, \tag{6}$$

where b_j is the *j*th largest of the arguments $a_1, a_2, \cdots, a_n$.

Note that the OWA operator is monotonic, commutative, bounded and idempotent. Furthermore, in (2004a), Yager developed the generalized OWA (GOWA) operator, which combines the generalized mean with the OWA operator. The GOWA operator is defined as follows:

Definition 4. A GOWA operator is mapping $GOWA: R^n \to R$ that has an associated weighting vector $\mathbf{w}$ of dimension *n* with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$, such that:

$$GOWA(a_1, a_2, \cdots, a_n) = \left(\sum_{j=1}^{n} w_j b_j^{\,r} \right)^{1/r}, \tag{7}$$

where b_j is the *j*th largest of a_i, and *r* is parameter such that $r \in (-\infty, \infty)$ and $r \neq 0$.

The GOWA operator is monotonic, commutative, bounded and idempotent (Yager 2004a). If we consider the possible values of the parameter *r* in the GOWA operator, we can obtain a group of particular cases. For examples, the OWA operator (Yager 1988), the ordered weighted geometric averaging (OWGA) operator (Chiclana *et al.* 2000; Xu, Da 2002b) and the ordered weighted harmonic averaging (OWHA) operator (Yager 2004a) are obtained as follows:

- The OWA operator is found if $r = 1$.
- The OWGA operator is obtained when $r \to 0$.
- The OWHA operator is formed when $r = -1$.

1.3. The C-OWA operator

The C-OWA operator was developed by Yager (2004b), which extends the OWA operator. It is defined as follows:

Definition 5. A C-OWA operator is mapping $f: M \to R^{+}$ associated with a basic unit interval monotonic (BUM) function Q, such that:

$$f_Q(\tilde{a}) = f_Q([\tilde{a}^L, \tilde{a}^U]) = \int_0^1 \frac{dQ(y)}{dy}(\tilde{a}^U - y(\tilde{a}^U - \tilde{a}^L))dy, \tag{8}$$

where $\tilde{a} = [\tilde{a}^L, \tilde{a}^U] \in M$, and M is the set of all nonnegative interval numbers.

If $\lambda = \int_0^1 Q(y)dy$ is the attitudinal character of Q, then a general formulation of $f_Q(\tilde{a})$ can be obtained as follows:

$$f_Q(\tilde{a}) = f_Q([\tilde{a}^L, \tilde{a}^U]) = \lambda \tilde{a}^U + (1-\lambda)\tilde{a}^L. \tag{9}$$

As can be seen, the C-OWA operator can be considered as an aggregation, guided by the function Q, in which the arguments to be aggregated are all values in the interval $[\tilde{a}^L, \tilde{a}^U]$. That is, the interval $[\tilde{a}^L, \tilde{a}^U]$ can be replaced by the aggregation $f_Q([\tilde{a}^L, \tilde{a}^U])$ with different Q. For convenience, throughout this paper, we denote the C-OWA operator f_Q by f_λ.

Note that other interesting generalizations of the C-OWA operator can be investigated by following references (Chen, Zhou 2011, 2012; Wu *et al.* 2009, 2010; Yager, Xu 2006; Zhou, Chen 2011).

1.4. Distance measure

Definition 6. Let A_1, A_2, A_3 be the elements or sets. A distance measure must accomplish the following properties:

- Nonnegativity: $D(A_1, A_2) \ge 0$.
- Commutativity: $D(A_1, A_2) = D(A_2, A_1)$.
- Reflexivity: $D(A_1, A_1) = 0$.
- Triangle inequality: $D(A_1, A_2) + D(A_2, A_3) \ge D(A_1, A_3)$.

Note that by using different cases of the function D, we are able to obtain different types of distance measure, such as the weighted Hamming distance (Merigó, Casanovas 2010), the weighted Euclidean distance measure (Merigó, Casanovas 2011a; Merigó, Gil-Lafuente 2007) and the ordered weighted distance (OWD) measure (Xu, Chen 2008a, 2008b).

In order to measure the deviation between any two IFVs $\alpha_1 = (\mu_{\alpha_1}, \nu_{\alpha_1})$ and $\alpha_2 = (\mu_{\alpha_2}, \nu_{\alpha_2})$, Xu (2007c, 2010c) defined the following distance.

Definition 7. Let $\alpha_1 = (\mu_{\alpha_1}, \nu_{\alpha_1})$ and $\alpha_2 = (\mu_{\alpha_2}, \nu_{\alpha_2})$ be two IFVs, then:

$$d(\alpha_1, \alpha_2) = \frac{1}{2}\left(|\mu_{\alpha_1} - \mu_{\alpha_2}| + |\nu_{\alpha_1} - \nu_{\alpha_2}|\right), \tag{10}$$

is called the intuitionistic fuzzy distance (IFD) between α_1 and α_2.

Motivated by the idea of the OWD distance, Zeng and Su (2011) developed the intuitionistic fuzzy ordered weighted distance (IFOWD) measure, which is defined as follows:

Definition 8. An IFOWD measure is mapping $IFOWD:\Omega\times\Omega\to R^{+}$ that has an associated weighting vector $\mathbf{w}$ of dimension n, such that $\sum_{j=1}^{n} w_j=1$ and $w_j\in[0,1]$, according to the following formula:

$$IFOWD(\boldsymbol{\alpha},\boldsymbol{\beta})=\left(\sum_{j=1}^{n} w_j\left(d(\alpha_{\sigma(j)},\beta_{\sigma(j)})\right)^{r}\right)^{1/r}, \quad (11)$$

where $\sigma(1),\sigma(2),\cdots,\sigma(n)$ is any permutation of $(1,2,\ldots,n)$, such that:

$$d(\alpha_{\sigma(j-1)},\beta_{\sigma(j-1)})\geq d(\alpha_{\sigma(j)},\beta_{\sigma(j)}),\ j=2,3,\cdots,n, \quad (12)$$

and $d(\alpha_j,\beta_j)$ is the distance between α_j and β_j determined by Definition 7. $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\cdots,\alpha_n)$ and $\boldsymbol{\beta}=(\beta_1,\beta_2,\cdots,\beta_n)$ are two vectors of IFVs, and the parameter $r>0$.

The IFOWD measure is monotonic, bounded, idempotent and commutative. And if we consider the possible values of the parameter r in the IFOWD measure, we can obtain a group of particular cases, such as the intuitionistic fuzzy ordered weighted Hamming distance (IFOWHD) measure, the intuitionistic fuzzy ordered weighted geometric distance (IFOWGD) measure and the intuitionistic fuzzy ordered weighted Euclidean distance (IFOWED) measure, which are obtained as follows:

- The IFOWHD measure is found if $r=1$.
- The IFOWGD measure is obtained if $r\to 0$.
- The IFOWED measure is formed when $r=2$.

2. Continuous intuitionistic fuzzy ordered weighted distance measure

In this section, we will introduce the continuous intuitionistic fuzzy ordered weighted distance (C-IFOWD) measure, which is distance measure that uses the continuous interval-valued intuitionistic fuzzy ordered weighted averaging (C-IVIFOWA) operator in the distance measure of IVIFVs.

2.1. The C-IFOWD measure

The C-IVIFOWA operator can be defined as follows:

Definition 9. A C-IVIFOWA operator is mapping $g:\Sigma\to\Omega$ associated with the BUM function Q, such that:

$$g_Q(\tilde{\alpha})=\left(\mu_{g_Q(\tilde{\alpha})},\nu_{g_Q(\tilde{\alpha})}\right)=\left(f_\lambda([\tilde{\mu}_{\tilde{\alpha}}^{L},\tilde{\mu}_{\tilde{\alpha}}^{U}]),f_\lambda([\tilde{\nu}_{\tilde{\alpha}}^{L},\tilde{\nu}_{\tilde{\alpha}}^{U}])\right), \quad (13)$$

where $\tilde{\alpha}=(\tilde{\mu}_{\tilde{\alpha}},\tilde{\nu}_{\tilde{\alpha}})=([\tilde{\mu}_{\tilde{\alpha}}^{L},\tilde{\mu}_{\tilde{\alpha}}^{U}],[\tilde{\nu}_{\tilde{\alpha}}^{L},\tilde{\nu}_{\tilde{\alpha}}^{U}])\in\Sigma$, f_Q is the C-OWA operator determined by Equation (8). Q is the BUM function $Q:[0,1]\to[0,1]$, which is monotonic with $Q(0)=0$ and $Q(1)=1$.

If $\lambda=\int_0^1 Q(y)dy$ is the attitudinal character of Q, then by Equation (9) we have Theorem 1 as follows:

Theorem 1. If λ is the attitudinal character of Q, then:

$$g_Q(\tilde{\alpha})=(\mu_{g_Q(\tilde{\alpha})},\nu_{g_Q(\tilde{\alpha})})=\left(\lambda\tilde{\mu}_{\tilde{\alpha}}^U+(1-\lambda)\tilde{\mu}_{\tilde{\alpha}}^L,\lambda\tilde{\nu}_{\tilde{\alpha}}^U+(1-\lambda)\tilde{\nu}_{\tilde{\alpha}}^L\right). \tag{14}$$

As we can see, $0\le\lambda\tilde{\mu}_{\tilde{\alpha}}^U+(1-\lambda)\tilde{\mu}_{\tilde{\alpha}}^L\le 1$, $0\le\lambda\tilde{\nu}_{\tilde{\alpha}}^U+(1-\lambda)\tilde{\nu}_{\tilde{\alpha}}^L\le 1$ and

$$\lambda\tilde{\mu}_{\tilde{\alpha}}^U+(1-\lambda)\tilde{\mu}_{\tilde{\alpha}}^L+\lambda\tilde{\nu}_{\tilde{\alpha}}^U+(1-\lambda)\tilde{\nu}_{\tilde{\alpha}}^L=\lambda(\tilde{\mu}_{\tilde{\alpha}}^U+\tilde{\nu}_{\tilde{\alpha}}^U)+(1-\lambda)(\tilde{\mu}_{\tilde{\alpha}}^L+\tilde{\nu}_{\tilde{\alpha}}^L)\le$$
$$\lambda(\tilde{\mu}_{\tilde{\alpha}}^U+\tilde{\nu}_{\tilde{\alpha}}^U)+(1-\lambda)(\tilde{\mu}_{\tilde{\alpha}}^U+\tilde{\nu}_{\tilde{\alpha}}^U)=\tilde{\mu}_{\tilde{\alpha}}^U+\tilde{\nu}_{\tilde{\alpha}}^U\le 1,$$

which means that $g_Q(\tilde{\alpha})$ is an IFV.

The main advantage of the C-IVIFOWA operator is that it can be used to transform the IVIFV $\tilde{\alpha}$ to the IFV $g_Q(\tilde{\alpha})$ by the controlled parameter λ, which can reduce uncertainty of the IVIFV $\tilde{\alpha}$. Note that we also denote the C-IVIFOWA operator g_Q by g_λ.

Definition 10. Let $\tilde{\alpha}_1=(\tilde{\mu}_{\tilde{\alpha}_1},\tilde{\nu}_{\tilde{\alpha}_1})=([\tilde{\mu}_{\tilde{\alpha}_1}^L,\tilde{\mu}_{\tilde{\alpha}_1}^U],[\tilde{\nu}_{\tilde{\alpha}_1}^L,\tilde{\nu}_{\tilde{\alpha}_1}^U])$ and $\tilde{\alpha}_2=(\tilde{\mu}_{\tilde{\alpha}_2},\tilde{\nu}_{\tilde{\alpha}_2})=([\tilde{\mu}_{\tilde{\alpha}_2}^L,\tilde{\mu}_{\tilde{\alpha}_2}^U],[\tilde{\nu}_{\tilde{\alpha}_2}^L,\tilde{\nu}_{\tilde{\alpha}_2}^U])$ be two IVIFVs and g be the C-IVIFOWA operator, then

$$\tilde{d}_\lambda(\tilde{\alpha},\tilde{\beta})=\frac{1}{2}\left(|\mu_{g_Q(\tilde{\alpha})}-\mu_{g_Q(\tilde{\beta})}|+|\nu_{g_Q(\tilde{\alpha})}-\nu_{g_Q(\tilde{\beta})}|\right), \tag{15}$$

is called the continuous intuitionistic fuzzy distance between $\tilde{\alpha}$ and $\tilde{\beta}$ based on the C-IVIFOWA operator, where $\tilde{\alpha},\tilde{\beta}\in\Sigma$ and $g_Q(\tilde{\alpha})$, $g_Q(\tilde{\beta})$ are determined by Equation (13).

With Equation (14), $\tilde{d}_\lambda(\tilde{\alpha},\tilde{\beta})$ can be expressed as:

$$\tilde{d}_\lambda(\tilde{\alpha},\tilde{\beta})=\frac{1}{2}\left(|\lambda(\tilde{\mu}_{\tilde{\alpha}}^U-\tilde{\mu}_{\tilde{\beta}}^U)+(1-\lambda)(\tilde{\mu}_{\tilde{\alpha}}^L-\tilde{\mu}_{\tilde{\beta}}^L)|+|\lambda(\tilde{\nu}_{\tilde{\alpha}}^U-\tilde{\nu}_{\tilde{\beta}}^U)+(1-\lambda)(\tilde{\nu}_{\tilde{\alpha}}^L-\tilde{\nu}_{\tilde{\beta}}^L)|\right). \tag{16}$$

Specially, if $\tilde{\alpha}$ and $\tilde{\beta}$ are two IFVs, then Equation (16) reduces to Equation (10).

From Definition 10, we can get the following theorem easily:

Theorem 2. Let $\tilde{\alpha}_1=(\tilde{\mu}_{\tilde{\alpha}_1},\tilde{\nu}_{\tilde{\alpha}_1})=([\tilde{\mu}_{\tilde{\alpha}_1}^L,\tilde{\mu}_{\tilde{\alpha}_1}^U],[\tilde{\nu}_{\tilde{\alpha}_1}^L,\tilde{\nu}_{\tilde{\alpha}_1}^U])$, $\tilde{\alpha}_2=(\tilde{\mu}_{\tilde{\alpha}_2},\tilde{\nu}_{\tilde{\alpha}_2})=([\tilde{\mu}_{\tilde{\alpha}_2}^L,\tilde{\mu}_{\tilde{\alpha}_2}^U],[\tilde{\nu}_{\tilde{\alpha}_2}^L,\tilde{\nu}_{\tilde{\alpha}_2}^U])$ and $\tilde{\alpha}_3=(\tilde{\mu}_{\tilde{\alpha}_3},\tilde{\nu}_{\tilde{\alpha}_3})=([\tilde{\mu}_{\tilde{\alpha}_3}^L,\tilde{\mu}_{\tilde{\alpha}_3}^U],[\tilde{\nu}_{\tilde{\alpha}_3}^L,\tilde{\nu}_{\tilde{\alpha}_3}^U])$ be three IVIFVs, then for a certain λ,

(1) Nonnegativity: $\tilde{d}_\lambda(\tilde{\alpha}_1,\tilde{\alpha}_2)\ge 0$.

(2) Commutativity: $\tilde{d}_\lambda(\tilde{\alpha}_1,\tilde{\alpha}_2)=\tilde{d}_\lambda(\tilde{\alpha}_2,\tilde{\alpha}_1)$.

(3) Reflexivity: $\tilde{d}_\lambda(\tilde{\alpha}_1,\tilde{\alpha}_1)=0$.

(4) Triangle inequality: $\tilde{d}_\lambda(\tilde{\alpha}_1,\tilde{\alpha}_2)+\tilde{d}_\lambda(\tilde{\alpha}_1,\tilde{\alpha}_3)\ge\tilde{d}_\lambda(\tilde{\alpha}_2,\tilde{\alpha}_3)$.

Let $\tilde{\boldsymbol{\alpha}}=(\tilde{\alpha}_1,\tilde{\alpha}_2,\cdots,\tilde{\alpha}_n)\in\Sigma^n$ and $\tilde{\boldsymbol{\beta}}=(\tilde{\beta}_1,\tilde{\beta}_2,\cdots,\tilde{\beta}_n)\in\Sigma^n$, then we can define the continuous intuitionistic fuzzy ordered weighted distance (C-IFOWD) measure as follows:

Definition 11. A C-IFOWD measure is mapping $C-IFOWD:\Sigma^n\times\Sigma^n\to R$ that has an associated weighting vector $\mathbf{w}$ of dimension n, such that $\sum_{j=1}^n w_j=1$ and $w_j\in[0,1]$, according to the following formula:

$$C-IFOWD(\tilde{\boldsymbol{\alpha}},\tilde{\boldsymbol{\beta}})=\left(\sum_{j=1}^n w_j(\tilde{d}_\lambda(\tilde{\alpha}_{\sigma(j)},\tilde{\beta}_{\sigma(j)}))^r\right)^{1/r}, \tag{17}$$

where $\sigma(1),\sigma(2),\cdots,\sigma(n)$ is any permutation of $(1,2,\ldots,n)$, such that:

$$\tilde{d}_{\lambda}(\tilde{\alpha}_{\sigma(j-1)},\tilde{\beta}_{\sigma(j-1)}) \geq \tilde{d}_{\lambda}(\tilde{\alpha}_{\sigma(j)},\tilde{\beta}_{\sigma(j)}),\ j=2,3,\cdots,n, \tag{18}$$

$\tilde{d}_{\lambda}(\tilde{\alpha}_j,\tilde{\beta}_j)$ is the continuous intuitionistic fuzzy distance between $\tilde{\alpha}_j$ and $\tilde{\beta}_j$ based on the C-IVIFOWA operator and parameter $r>0$.

Obviously, if all $\tilde{\alpha}_j$ and $\tilde{\beta}_j$ are given in the form of IFVs, then the C-IFOWD measure reduces to the IFOWD measure (Zeng, Su 2011).

Example 1. Let $\tilde{\boldsymbol{\alpha}} = (([0.2,0.3],[0.4,0.5]),([0.6,0.7],[0.2,0.3]),([0.4,0.5],[0.2,0.4]),([0.5, 0.6],[0.1, 0.3])$ and $\tilde{\boldsymbol{\beta}} = (([0.7,0.8],[0.1,0.2]),([0.1,0.3],[0.5,0.6]),([0.5, 0.7]\ [0.2, 0.3]),([0.3, 0.4],[0.5, 0.6]))$ be two vectors of IVIFVs. Assume the weighting vector of the ordered positions of the distances $\tilde{d}_{\lambda}(\tilde{\alpha}_j,\tilde{\beta}_j)$ $(j=1,\cdots,4)$: $w=(0.2,0.4,0.3,0.1)$. If we assume that $Q(y)=y^2$, then $\mu=\int_0^1 Q(y)dy=1/3$. By Equation (16), we can get:

$$\tilde{d}_{\lambda}(\tilde{\alpha}_1,\tilde{\beta}_1)=\frac{1}{2}\left(|(0.3-0.8)/3+2(0.2-0.7)/3|+|(0.5-0.2)/3+2(0.4-0.1)/3|\right)=0.4;$$

$$\tilde{d}_{\lambda}(\tilde{\alpha}_2,\tilde{\beta}_2)=\frac{1}{2}\left(|(0.7-0.3)/3+2(0.6-0.1)/3|+|(0.3-0.6)/3+2(0.2-0.5)/3|\right)=0.38;$$

$$\tilde{d}_{\lambda}(\tilde{\alpha}_3,\tilde{\beta}_3)=\frac{1}{2}\left(|(0.5-0.7)/3+2(0.4-0.5)/3|+|(0.4-0.3)/3+2(0.2-0.2)/3|\right)=0.08;$$

$$\tilde{d}_{\lambda}(\tilde{\alpha}_4,\tilde{\beta}_4)=\frac{1}{2}\left(|(0.6-0.4)/3+2(0.5-0.3)/3|+|(0.3-0.6)/3+2(0.1-0.5)/3|\right)=0.28.$$

Thus,

$\tilde{d}_{\lambda}(\tilde{\alpha}_{\sigma(1)},\tilde{\beta}_{\sigma(1)})=0.4$, $\tilde{d}_{\lambda}(\tilde{\alpha}_{\sigma(2)},\tilde{\beta}_{\sigma(2)})=0.38$;

$\tilde{d}_{\lambda}(\tilde{\alpha}_{\sigma(3)},\tilde{\beta}_{\sigma(3)})=0.28$, $\tilde{d}_{\lambda}(\tilde{\alpha}_{\sigma(4)},\tilde{\beta}_{\sigma(4)})=0.08$.

By Equation (17), we can obtain the distances corresponding to some special cases of the parameter *r*, which are shown in Table 1.

Table 1. Aggregation results

r	$\to 0$	0.1	1	2	3	4
$C-IFOWD(\tilde{\boldsymbol{\alpha}},\tilde{\boldsymbol{\beta}})$	0.2980	0.3029	0.3240	0.3375	0.3459	0.3517
r	5	6	7	8	9	10
$C-IFOWD(\tilde{\boldsymbol{\alpha}},\tilde{\boldsymbol{\beta}})$	0.3562	0.3598	0.3627	0.3652	0.3673	0.3691

From Table 1, the aggregation result $C-IFOWD(\tilde{\boldsymbol{\alpha}},\tilde{\boldsymbol{\beta}})$ increases as the parameter *r* steadily increases.

As we can see, the characteristic of the C-IFOWD measure is that it combines the GOWA operator with the distance measure based on the C-IVIFOWA operator. The principal advantages of the C-IFOWD measure are that it is not only a generalization of some widely used distance measure, but also it can deal with interval deviations in aggregation on IVIFVs by using the controlled parameter, which makes the C-IFOWD measure very suitable to wide range situations, such as decision making, engineering and economics.

Note that following Xu and Da (2002a), it is possible to distinguish between the descending C-IFOWD (DC-IFOWD) measure and the ascending C-IFOWD (AC-IFOWD) measure by using $w_j = w^*_{n-j+1}$, where w_j and w^*_{n-j+1} are the *j*th weighting coefficient of the weighting vector in the DC-IFOWD measure and the AC-IFOWD measure, respectively.

2.2. Properties of the C-IFOWD measure

The C-IFOWD measure is monotonic, idempotent, bounded, commutative, nonnegative and reflexive. These desirable properties are shown with the following theorem.

Theorem 3. Let $\tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) \in \Sigma^n, \tilde{\beta} = (\tilde{\beta}_1, \tilde{\beta}_2, \cdots, \tilde{\beta}_n) \in \Sigma^n$ and $\tilde{\gamma} = (\tilde{\gamma}_1, \tilde{\gamma}_2, \cdots, \tilde{\gamma}_n) \in \Sigma^n$, then:

(1) (Nonnegativity) $C - IFOWD(\tilde{\alpha}, \tilde{\beta}) \geq 0.$ (19)

(2) (Reflexivity) $C - IFOWD(\tilde{\alpha}, \tilde{\alpha}) = 0.$ (20)

(3) (Commutativity – distance measure) $C - IFOWD(\tilde{\alpha}, \tilde{\beta}) = C - IFOWD(\tilde{\beta}, \tilde{\alpha}).$ (21)

(4) (Commutativity – GOWA aggregation) If $(d_\lambda(\hat{\alpha}_1, \hat{\beta}_1),\ d_\lambda(\hat{\alpha}_2, \hat{\beta}_2), \cdots, d_\lambda(\hat{\alpha}_n, \hat{\beta}_n))$ is a permutation of $(d_\lambda(\tilde{\alpha}_1, \tilde{\beta}_1), d_\lambda(\tilde{\alpha}_2, \tilde{\beta}_2), \cdots, d_\lambda(\tilde{\alpha}_n, \tilde{\beta}_n))$, then

$$C - IFOWD(\hat{\alpha}, \hat{\beta}) = C - IFOWD(\tilde{\alpha}, \tilde{\beta}), \tag{22}$$

where $\hat{\alpha} = (\hat{\alpha}_1, \hat{\alpha}_2, \cdots, \hat{\alpha}_n) \in \Sigma^n$ and $\hat{\beta} = (\hat{\beta}_1, \hat{\beta}_2, \cdots, \hat{\beta}_n) \in \Sigma^n$.

(5) (Idempotency) If $\tilde{d}_\lambda(\tilde{\alpha}_j, \tilde{\beta}_j) = d_0$ for all *j*, then

$$C - IFOWD(\tilde{\alpha}, \tilde{\beta}) = d_0. \tag{23}$$

(6) (Boundedness) If $\max_j \tilde{d}_\lambda(\tilde{\alpha}_j, \tilde{\beta}_j) = d_{\max}$ and $\min_j \tilde{d}_\lambda(\tilde{\alpha}_j, \tilde{\beta}_j) = d_{\min}$ for a certain λ, then

$$d_{\min} \leq C - IFOWD(\tilde{\alpha}, \tilde{\beta}) \leq d_{\max}. \tag{24}$$

(7) (Monotonicity – parameter *r*) Assume that $C - IFOWD_r(\tilde{\alpha}, \tilde{\beta})$ is the C-IFOWD measure. If $r_1 \geq r_2$, then

$$C - IFOWD_{r_1}(\tilde{\alpha}, \tilde{\beta}) \geq C - IFOWD_{r_2}(\tilde{\alpha}, \tilde{\beta}). \tag{25}$$

(8) (Monotonicity – distance measure) If $\tilde{d}_\lambda(\tilde{\alpha}_j, \tilde{\beta}_j) \leq \tilde{d}_\lambda(\tilde{\alpha}_j, \tilde{\gamma}_j)$ with a certain λ for all *j*, then

$$C - IFOWD(\tilde{\alpha}, \tilde{\beta}) \leq C - IFOWD(\tilde{\alpha}, \tilde{\gamma}), \tag{26}$$

where $\tilde{d}_\lambda$ is the distance based on the C-IVIFOWA operator.

Proof. We can get the results (1–4) by Theorem 2 and Definition 11 immediately, and we focus on proving results (5–6) as follows:

(5) If $\tilde{d}_\lambda(\tilde{\alpha}_j, \tilde{\beta}_j) = d_0$ for all *j*, then

$$C - IFOWD(\tilde{\alpha}, \tilde{\beta}) = \left(\sum_{j=1}^{n} w_j (d_0)^r \right)^{1/r} = \left((d_0)^r \sum_{j=1}^{n} w_j \right)^{1/r} = d_0.$$

(6) If $\max_j \tilde{d}_\lambda(\tilde{\alpha}_j, \tilde{\beta}_j) = d_{\max}$ and $\min_j \tilde{d}_\lambda(\tilde{\alpha}_j, \tilde{\beta}_j) = d_{\min}$ for a certain λ, then according to boundedness of the GOWA operator (Yager 2004a), we get

$$d_{\min} \le C-IFOWD(\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}) \le d_{\max}.$$

(7) $C-IFOWD_{r_1}(\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}) = \left(\sum_{j=1}^{n} w_j (\tilde{d}_\lambda(\tilde{\alpha}_{\sigma(j)}, \tilde{\beta}_{\sigma(j)}))^{r_1}\right)^{1/r_1}$, and

$$C-IFOWD_{r_2}(\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}) = \left(\sum_{j=1}^{n} w_j (\tilde{d}_\lambda(\tilde{\alpha}_{\sigma(j)}, \tilde{\beta}_{\sigma(j)}))^{r_2}\right)^{1/r_2}.$$

If $r_1 \ge r_2$, then by monotonicity of the GOWA operator, then

$$C-IFOWD_{r_1}(\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}) \ge C-IFOWD_{r_2}(\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}).$$

(8) Let $C-IFOWD(\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}) = \left(\sum_{j=1}^{n} w_j (\tilde{d}_\lambda(\tilde{\alpha}_{\sigma(j)}, \tilde{\beta}_{\sigma(j)}))^{r}\right)^{1/r}$

and

$$C-IFOWD(\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\gamma}}) = \left(\sum_{j=1}^{n} w_j (\tilde{d}_\lambda(\tilde{\alpha}_{\sigma(j)}, \tilde{\gamma}_{\sigma(j)}))^{r}\right)^{1/r}.$$

If $\tilde{d}_\lambda(\tilde{\alpha}_j, \tilde{\beta}_j) \le \tilde{d}_\lambda(\tilde{\alpha}_j, \tilde{\gamma}_j)$ with a certain λ for all *j*, then according to the monotonicity of the GOWA operator, we obtain:

$$C-IFOWD(\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}) \le C-IFOWD(\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\gamma}}).$$

The theorem is proved.

2.3. Families of the C-IFOWD measure

By using different cases of the parameter *r* and the weighting vector in the C-IFOWD measure, we are able to obtain different types of distance measure, such as the continuous intuitionistic fuzzy ordered weighted Hamming distance (C-IFOWHD) measure, the continuous intuitionistic fuzzy ordered weighted Euclidean distance (C-IFOWED) measure, the continuous intuitionistic fuzzy ordered weighted geometric distance (C-IFOWGD) measure, the Median C-IFOWD measure, the Olympic C-IFOWD measure, etc.

Remark 1. If $r = 1$, then the C-IFOWD measure reduces to the C-IFOWHD measure:

$$C-IFOWHD(\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}) = \sum_{j=1}^{n} w_j \tilde{d}_\lambda(\tilde{\alpha}_{\sigma(j)}, \tilde{\beta}_{\sigma(j)}). \tag{27}$$

And if $r = 2$, then the C-IFOWD measure becomes the C-IFOWED measure:

$$C-IFOWED(\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}) = \left(\sum_{j=1}^{n} w_j (\tilde{d}_\lambda(\tilde{\alpha}_{\sigma(j)}, \tilde{\beta}_{\sigma(j)}))^{2}\right)^{1/2}. \tag{28}$$

If $r \to 0$, then we get that:

$$C-IFOWGD(\tilde{\alpha},\tilde{\beta}) = \prod_{j=1}^{n} (\tilde{d}_\lambda(\tilde{\alpha}_{\sigma(j)}, \tilde{\beta}_{\sigma(j)}))^{w_j}, \tag{29}$$

which is called the continuous intuitionistic fuzzy ordered weighted geometric distance (C-IFOWGD) measure.

Remark 2. The continuous intuitionistic fuzzy maximum distance (C-IFMAXD) measure, the continuous intuitionistic fuzzy minimum distance (C-IFMIND) measure, the Step C-IFOWD measure, the continuous intuitionistic fuzzy normalized distance (C-IFND) measure, the continuous intuitionistic fuzzy normalized Hamming distance (C-IFNHD) measure, the continuous intuitionistic fuzzy normalized geometric distance (C-IFNGD) measure, the continuous intuitionistic fuzzy normalized Euclidean distance (C-IFNED) measure and the median C-IFOWD measure are obtained as follows:

- The C-IFMAXD measure is found if $w_1 = 1$ and $w_j = 0$ for all $j \neq 1$.
- The C-IFMIND measure is formed if $w_n = 1$ and $w_j = 0$ for all $j \neq n$.
- Generally, if $w_k = 1$ and $w_j = 0$ for all $j \neq k$, then we get the Step C-IFOWD measure.
- The C-IFND measure is found when $w_j = 1/n$ for all j. Specially, if $r = 1$, we get the C-IFNHD measure, and if $r \to 0$, we obtain the C-IFNGD measure. If $r = 2$, we have the C-IFNED measure.
- If $w_{(n+1)/2} = 1$, $w_j = 0$ for $j \neq (n+1)/2$, n is odd, or $w_{n/2} = w_{n/2+1} = 1/2$, $w_j = 0$ for $j \neq n/2, n/2+1$, then we get the Median C-IFOWD measure.

Remark 3. Similar to the literatures (Merigó 2011; Merigó, Gil-Lafuente 2009, 2011b; Yager 1993, 2007), we can obtain a lot of families of C-IFOWD measure such as:

- The Olympic C-IFOWD measure ($w_1 = w_n = 0$ and $w_j = 1/(n-2)$ for $j \neq 1, n$).
- The general Olympic C-IFOWD measure ($w_j = 0$ for $j = 1, 2, ..., k, n, n-1, ..., n-k+1$; and for all others $w_j = 1/(n-2k)$, where $k < n/2$).
- The Step C-IFOWD measure ($w_k = 1$ and $w_j = 0$ for all $j \neq k$).
- The Window C-IFOWD measure ($w_j = 1/m$ for $k \leq j \leq k+m-1$, and $w_j = 0$ for $j \geq k+m$ and $j < k$).
- The generalized S C-IFOWD measure ($w_k = (1-(\alpha+\beta))/n + \alpha$, $w_t = (1-(\alpha+\beta))/n + \beta$ and $w_j = (1-(\alpha+\beta))/n$ for all $j \neq k, t$, where $a_k = \max_i\{a_i\}$, $a_t = \min_i\{a_i\}$ and $\alpha+\beta \leq 1$ with $\alpha, \beta \in [0,1]$).
- The Centered C-IFOWD measure is symmetric, strongly decaying and inclusive.

Remark 4. Using a similar methodology, we can develop numerous other families of the C-IFOWD measure following Merigó and Gil-Lafuente (2009), Xu (2006b), Yager (1993, 1996, 2009), Yu and Xu (2013).

2.4. Extensions of the C-IFOWD measure

Following Liu (2010), Merigó (2011), Mesiar and Pap (2008), it is possible to develop an extension of the C-IFOWD measure by using the quasi-arithmetic means instead of the generalized means. This result is the quasi-arithmetic C-IFOWD (Quasi-C-IFOWD) measure, which can be defined as follows:

Definition 12. A Quasi-C-IFOWD measure is mapping $Quasi-C-IFOWD$: $\Sigma^n \times \Sigma^n \to R$ that has an associated weighting vector $\mathbf{w}$ of dimension n, such that $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$, according to the following formula:

$$Quasi-C-IFOWD(\tilde{\alpha}, \tilde{\beta}) = h^{-1}\left(\sum_{j=1}^{n} w_j h(\tilde{d}_\lambda(\tilde{\alpha}_{\sigma(j)}, \tilde{\beta}_{\sigma(j)}))\right), \tag{30}$$

where h is a strictly continuous monotonic function; $\sigma(1), \sigma(2), \cdots, \sigma(n)$ is any permutation of $(1,2,\ldots,n)$, such that $\tilde{d}_\lambda(\tilde{\alpha}_{\sigma(j-1)}, \tilde{\beta}_{\sigma(j-1)}) \geq \tilde{d}_\lambda(\tilde{\alpha}_{\sigma(j)}, \tilde{\beta}_{\sigma(j)})$, $j = 2,3,\cdots,n$, $\tilde{d}_\lambda(\tilde{\alpha}_j, \tilde{\beta}_j)$ is the distance between $\tilde{\alpha}_j$ and $\tilde{\beta}_j$ based on the C-IFOWA operator and λ is the attitudinal character of BUM function Q.

Note that if $h(x) = x^r$, then we get the C-IFOWD measure. The main advantage of this measure is that it provides a more complete generalization including a lot of special cases that are not included in the C-IFOWD measure.

Another important extension is the infinitary C-IFOWD (∞-C-IFOWD) measure, which uses infinitary aggregation operators (Mesiar, Pap 2008). It can be defined as follows:

Definition 13. An ∞-C-IFOWD measure is mapping $\infty - C - IFOWD: \Sigma^\infty \times \Sigma^\infty \to R$ that has an associated weighting vector $\mathbf{w}$ of dimension n such that $\sum_{j=1}^{\infty} w_j = 1$ and $w_j \in [0,1]$, according to the following formula:

$$\infty - C - IFOWD(\tilde{\alpha}, \tilde{\beta}) = \left(\sum_{j=1}^{\infty} w_j (\tilde{d}_\lambda(\tilde{\alpha}_{\sigma(j)}, \tilde{\beta}_{\sigma(j)}))^r\right)^{1/r}, \tag{31}$$

where $\sigma(1), \sigma(2), \cdots, \sigma(n), \cdots$ is any permutation of $(1,2,\ldots,n,\cdots)$, such that $\tilde{d}_\lambda(\tilde{\alpha}_{\sigma(j-1)}, \tilde{\beta}_{\sigma(j-1)}) \geq \tilde{d}_\lambda(\tilde{\alpha}_{\sigma(j)}, \tilde{\beta}_{\sigma(j)})$, j = 2, 3,..., n, ..., $\tilde{d}_\lambda(\tilde{\alpha}_j, \tilde{\beta}_j)$ is the distance between $\tilde{\alpha}_j$ and $\tilde{\beta}_j$ based on the C-IFOWA operator and λ is the attitudinal character of BUM function Q.

Note that a similar extension could be developed by combining the Quasi-C-IFOWD measure with the ∞-C-IFOWD measure, then we can obtain the ∞-Quasi-C-IFOWD measure. Note also that other interesting generalizations can be investigated following Merigó (2008, 2011), Merigó and Casanovas (2011c), Pereira and Ribeiro (2003), such as the mixture C-IFOWD measure, the heavy C-IFOWD measure, the C-IFOWD weighted average measure, etc.

3. Multiple attributes group decision making with the C-IFOWD measure

The C-IFOWD measure is applicable in a wide range of situations, such as decision making, economics, statistics and engineering. In this section, we develop an application of the C-IFOWD measure in a multiple attributes group decision making problem.

Consider a multiple attributes group decision making problem. Let $X = \{x_1, x_2, \cdots, x_m\}$ be a discrete set of m feasible alternatives, and $U = \{u_1, u_2, \cdots, u_n\}$ be a finite set of attributes. Let $E = \{e_1, e_2, \cdots, e_t\}$ be the set of decision makers, and $\mathbf{v} = (v_1, v_2, \cdots, v_t)$ be the weighting vector of decision makers satisfying $v_k \in [0,1]$ and $\sum_{k=1}^{t} v_k = 1$. Assume that each decision maker provides his own decision matrix $\tilde{A}^{(k)} = (\tilde{a}_{ij}{}^{(k)})_{m\times n}$, in which $\tilde{a}_{ij}{}^{(k)} \in \Sigma$ is given by the decision maker $e_k \in E$, for the alternative $x_i \in X$ with respect to the attribute $u_j \in U$. Let $\mathbf{w} = (w_1, w_2, \cdots, w_n)$ be the weighting vector of attributes satisfying $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$.

The process with the C-IFOWD measure in multiple attribute group decision making involves the following steps:

Step 1. Form the ideal alternative by giving the ideal levels of each characteristic, which is shown in Table 2, where $x^{(k)}$ is the ideal alternative and $\tilde{y}_j{}^{(k)}$ is the jth ideal characteristic of $x^{(k)}$.

Table 2. Ideal alternative

	u_1	u_2	$\cdots$	u_j	$\cdots$	u_n
$x^{(k)}$	$\tilde{y}_1{}^{(k)}$	$\tilde{y}_2{}^{(k)}$	$\cdots$	$\tilde{y}_j{}^{(k)}$	$\cdots$	$\tilde{y}_n{}^{(k)}$

Step 2. Calculate the distance of each preference value $\tilde{a}_{ij}{}^{(k)}$ provided by the decision maker e_k and his/her ideal preference value $\tilde{y}_j{}^{(k)}$ by Equation (32):

$$\tilde{d}_\lambda(\tilde{a}_{ij}^{(k)}, \tilde{y}_j^{(k)}) = \frac{1}{2}\Bigg(\left| \lambda(\tilde{\mu}^U_{\tilde{a}_{ij}^{(k)}} - \tilde{\mu}^U_{\tilde{y}_j^{(k)}}) + (1-\lambda)(\tilde{\mu}^L_{\tilde{a}_{ij}^{(k)}} - \tilde{\mu}^L_{\tilde{y}_j^{(k)}}) \right| + \left| \lambda(\tilde{v}^U_{\tilde{a}_{ij}^{(k)}} - \tilde{v}^U_{\tilde{y}_j^{(k)}}) + (1-\lambda)(\tilde{v}^L_{\tilde{a}_{ij}^{(k)}} - \tilde{v}^L_{\tilde{y}_j^{(k)}}) \right| \Bigg),\ i = 1,2,\cdots,m,\ j = 1,2,\cdots,n,\ k = 1,2,\cdots,t, \tag{32}$$

where $\lambda = \int_0^1 Q(y)dy$ is the attitudinal character of Q.

Step 3. Utilize the C-IFOWD measure:

$$\tilde{r}_{ik} = C-IFOWD(\tilde{\mathbf{a}}_i^{(k)}, \tilde{\mathbf{y}}^{(k)}) = \left(\sum_{j=1}^{n} w_j (\tilde{d}_\lambda(\tilde{a}_{ij}^{(k)}, \tilde{y}_j^{(k)}))^r \right)^{1/r},\ i = 1,2,\cdots,m,\ k = 1,2,\cdots,t, \tag{33}$$

to aggregate all distance into a collective distance matrix $\tilde{R} = (\tilde{r}_{ik})_{m\times t}$, where $\tilde{\mathbf{a}}_i^{(k)} = (\tilde{a}_{i1}^{(k)}, \tilde{a}_{i2}^{(k)}, \cdots, \tilde{a}_{in}^{(k)})$.

Step 4. Utilize the GOWA operator:

$$\tilde{r}_i = GOWA(\tilde{r}_{i1}, \tilde{r}_{i2}, \cdots, \tilde{r}_{it}),\ i = 1,2,\cdots,m,$$

to derive the collective overall preference value $\tilde{r}_i$ of the alternative x_i.

Step 5. Rank the collective overall preference values $\tilde{r}_i$ ($i = 1,2,\cdots,m$) in ascending order.

Step 6. Rank all the alternatives x_i ($i = 1,2,\cdots,m$) and select the best one(s) in accordance with the collective overall preference values $\tilde{r}_i$ ($i = 1,2,\cdots,m$). Note that the best choice is the one with the lowest distance.

Step 7. End.

4. Illustrative example

In the following, we develop a brief illustrative example of the new approach in a group decision making problem under interval-valued intuitionistic fuzzy environment. We study a human resource management problem where a university wants to introduce oversea outstanding teachers (adapted from Yu *et al.* 2012). The university has brought together a group of decision makers. The group is constituted by three persons including university president e_1, dean of management school e_2 and human resource officer e_3. After careful review of the information, they made strict evaluation for five candidates x_i ($i=1,2,3,4,5$) and summarized the abilities of candidates with four aspects $U=\{u_1,u_2,u_3,u_4\}$:

- u_1 : Namely morality.
- u_2 : Research capability.
- u_3 : Teaching skill.
- u_4 : Education background.

Three decision makers evaluate the candidates x_i ($i=1,2,3,4,5$) with respect to the attributes u_j ($j=1,2,3,4$) and construct three interval-valued intuitionistic fuzzy decision matrices $\tilde{A}^{(k)}=(\tilde{a}_{ij}^{(k)})_{5\times 4}$ ($k=1,2,3$), which are shown in Tables 3, 4, 5.

Table 3. Interval-valued intuitionistic fuzzy decision matrix $\tilde{R}^{(1)}$

	u_1	u_2	u_3	u_4
x_1	([0.6,0.8],[0.1,0.2])	([0.2,0.4],[0.4,0.5])	([0.6,0.7],[0.2,0.3])	([0.4,0.5],[0.2,0.4])
x_2	([0.4,0.7],[0,0.1])	([0.5,0.7],[0.1,0.2])	([0.7,0.8],[0.1,0.2])	([0.7,0.8],[0.1,0.2])
x_3	([0.3,0.7],[0.2,0.3])	([0.2,0.4],[0.4,0.5])	([0.1,0.4],[0.4,0.5])	([0.3,0.4],[0.4,0.6])
x_4	([0.7,0.8],[0.1,0.2])	([0.2,0.3],[0.4,0.6])	([0.6,0.8],[0,0.2])	([0.6,0.8],[0,0.2])
x_5	([0.5,0.6],[0.3,0.4])	([0.7,0.8],[0,0.1])	([0.2,0.4],[0.4,0.5])	([0.1,0.3],[0.4,0.6])

Table 4. Interval-valued intuitionistic fuzzy decision matrix $\tilde{R}^{(2)}$

	u_1	u_2	u_3	u_4
x_1	([0.2,0.4],[0.4,0.5])	([0.6,0.7],[0.1,0.2])	([0.5,0.7],[0.1,0.2])	([0.5,0.7],[0.1,0.2])
x_2	([0.6,0.8],[0,0.2])	([0.2,0.3],[0.4,0.6])	([0.7,0.8],[0.1,0.2])	([0.2,0.4],[0.4,0.5])
x_3	([0.1,0.4],[0.4,0.5])	([0.8,0.9],[0,0.1])	([0.1,0.4],[0.2,0.5])	([0.4,0.7],[0.2,0.3])
x_4	([0.6,0.8],[0,0.2])	([0.3,0.8],[0,0.1])	([0.2,0.3],[0.4,0.6])	([0.6,0.7],[0.2,0.3])
x_5	([0.2,0.4],[0.5,0.6])	([0.6,0.7],[0.2,0.3])	([0.6,0.8],[0,0.2])	([0.1,0.4],[0.3,0.5])

Table 5. Interval-valued intuitionistic fuzzy decision matrix $\tilde{R}^{(3)}$

	u_1	u_2	u_3	u_4
x_1	([0.2,0.4],[0.4,0.5])	([0.2,0.4],[0.4,0.5])	([0.4,0.7],[0,0.1])	([0.7,0.9],[0,0.1])
x_2	([0.2,0.3],[0.4,0.6])	([0.2,0.3],[0.4,0.6])	([0.6,0.7],[0.2,0.3])	([0.5,0.7],[0.1,0.2])
x_3	([0.7,0.9],[0,0.1])	([0.3,0.4],[0.4,0.5])	([0.1,0.3],[0.3,0.5])	([0.2,0.4],[0.4,0.5])
x_4	([0.3,0.8],[0.1,0.2])	([0.1,0.2],[0.4,0.6])	([0.2,0.3],[0.4,0.5])	([0.3,0.4],[0.4,0.6])
x_5	([0.7,0.8],[0,0.2])	([0.3,0.8],[0,0.1])	([0.4,0.7],[0.2,0.3])	([0.6,0.7],[0,0.2])

With this information, we can use the proposed decision making method to get the ranking of the candidates. The following steps are involved:

Step 1. According to the objectives of the university, each expert establishes his/her own ideal strategy. The results are shown in Table 6.

Table 6. Ideal strategy

	u_1	u_2	u_3	u_4
e_1	([0.4,0.6],[0.2,0.4])	([0.3,0.5],[0.3,0.5])	([0.5,0.6],[0.3,0.4])	([0.3,0.5],[0.3,0.5])
e_2	([0.5,0.7],[0.2,0.3])	([0.5,0.7],[0.2,0.3])	([0.5,0.7],[0.1,0.3])	([0.4,0.6],[0.2,0.4])
e_3	([0.5,0.6],[0.2,0.3])	([0.4,0.6],[0.2,0.4])	([0.5,0.7],[0.2,0.3])	([0.4,0.5],[0.3,0.4])

Step 2. Calculate the distance of each preference value $\tilde{a}_{ij}^{(k)}$ provided by the decision maker e_k and his/her ideal preference value $\tilde{y}_j^{(k)}$ with Equation (32), and the results are shown in Tables 7–9, where $Q(y) = y^2$.

Table 7. Distance matrix – expert 1

	u_1	u_2	u_3	u_4
x_1	0.1667	0.0833	0.1000	0.0833
x_2	0.1333	0.2167	0.2000	0.3000
x_3	0.0330	0.0833	0.2167	0.0667
x_4	0.2000	0.1167	0.2000	0.3000
x_5	0.0667	0.3500	0.1833	0.1500

Table 8. Distance matrix – expert 2

	u_1	u_2	u_3	u_4
x_1	0.2500	0.0833	0.0167	0.1167
x_2	0.1333	0.2833	0.1000	0.1833
x_3	0.2833	0.2333	0.2500	0.0333
x_4	0.1333	0.1500	0.3167	0.1000
x_5	0.3000	0.0333	0.1000	0.1833

Table 9. Distance matrix – expert 3

	u_1	u_2	u_3	u_4
x_1	0.2333	0.1833	0.1333	0.3167
x_2	0.2667	0.2167	0.0333	0.1667
x_3	0.2167	0.1500	0.2667	0.1333
x_4	0.0833	0.2667	0.2667	0.1167
x_5	0.1833	0.1167	0.0333	0.2333

Step 3. Utilize the C-IFOWD measure to aggregate the whole distance into a collective distance matrix $\tilde{R} = (\tilde{r}_{ik})_{5\times 3}$, where $r = 1.5$, $\mathbf{w} = (0.3, 0.2, 0.4, 0.1)$ and

$$\tilde{R} = \begin{pmatrix} 0.1145 & 0.1452 & 0.2327 \\ 0.2297 & 0.1912 & 0.1998 \\ 0.1219 & 0.2378 & 0.2002 \\ 0.2253 & 0.1894 & 0.1966 \\ 0.2194 & 0.1829 & 0.1634 \end{pmatrix}.$$

Step 4. Utilize the GOWA operator to derive the collective overall preference value $\tilde{r}_i$ of the alternative x_i:

$$\tilde{r}_1 = 0.1749,\ \tilde{r}_2 = 0.2095,\ \tilde{r}_3 = 0.1949,\ \tilde{r}_4 = 0.2062,\ \tilde{r}_5 = 0.1924.$$

Note that we assume that the parameter r in the GOWA operator is equal to the parameter r in the C-IFOWD measure, and $\mathbf{v} = (0.4, 0.3, 0.3)$ is the weighting vector of three experts.

Step 5. Rank the collective overall preference values $\tilde{r}_i$ ($i = 1, 2, \cdots, 5$) in ascending order:

$$\tilde{r}_1 < \tilde{r}_5 < \tilde{r}_3 < \tilde{r}_4 < \tilde{r}_2.$$

Step 6. Rank all the alternatives x_i ($i = 1, 2, \cdots, 5$) and select the best one(s) in accordance with the collective overall preference values $\tilde{r}_i$ ($i = 1, 2, \cdots, 5$):

$$x_1 \succ x_5 \succ x_3 \succ x_4 \succ x_2.$$

As we can see, the best one is x_5. That is to say, optimal alternative for the university is the first candidate.

Furthermore, in order to analyze how the different particular cases of the C-IFOWD measure have affection for the aggregation results, in this example, we consider the C-IFMAXD measure, the C-IFMIND measure, the C-IFOWD measure, the AC-IFOWD measure, the C-IFND measure, the C-IFNHD measure, the C-IFNGD measure, the C-IFNED measure, the Median C-IFOWD measure, the Step C-IFOWD measure ($k = 3$), the Olympic C-IFOWD measure and the continuous intuitionistic fuzzy Hurwicz distance measure ($\alpha = 0.4$). The results are shown in Tables 10 and 11.

Table 10. Aggregation results 1

	C-IFOWD	AC-IFOWD	C-IFND	C-IFNHD	C-IFNGD	C-IFNED
x_1	0.1749	0.1479	0.1653	0.1313	0.1418	0.1841
x_2	0.2095	0.1799	0.1963	0.2042	0.1904	0.2143
x_3	0.1949	0.1656	0.1802	0.1852	0.1583	0.2026
x_4	0.2062	0.1793	0.1944	0.2017	0.1868	0.2105
x_5	0.1924	0.1534	0.1777	0.1813	0.1565	0.2027

Table 11. Aggregation results 2

	C-IFMAXD	C-IFMIND	Median	Step	Olympic	Hurwicz
x_1	0.2556	0.0907	0.1467	0.1281	0.1467	0.1667
x_2	0.2852	0.0982	0.1895	0.1711	0.1895	0.1847
x_3	0.2591	0.0812	0.1819	0.1661	0.1819	0.1635
x_4	0.2553	0.1072	0.1796	0.1527	0.1796	0.1782
x_5	0.3020	0.0481	0.1560	0.1259	0.1560	0.1742

We can establish an ordering of the candidates for each special distance measure. The results are shown in Table 12. Note that "$\succ$" means "preferred to".

Table 12. Ordering of the candidates

	Ordering		Ordering
C-IFOWD	$x_1 \succ x_5 \succ x_3 \succ x_4 \succ x_2$	C-IFMAXD	$x_4 \succ x_1 \succ x_3 \succ x_2 \succ x_5$
AC-IFOWD	$x_1 \succ x_5 \succ x_3 \succ x_4 \succ x_2$	C-IFMIND	$x_5 \succ x_3 \succ x_1 \succ x_2 \succ x_4$
C-IFND	$x_1 \succ x_5 \succ x_3 \succ x_4 \succ x_2$	Median C-IFOWD	$x_1 \succ x_5 \succ x_4 \succ x_3 \succ x_2$
C-IFNHD	$x_1 \succ x_5 \succ x_3 \succ x_4 \succ x_2$	Step C-IFOWD	$x_1 \succ x_5 \succ x_4 \succ x_3 \succ x_2$
C-IFNGD	$x_1 \succ x_5 \succ x_3 \succ x_4 \succ x_2$	Olympic C-IFOWD	$x_1 \succ x_5 \succ x_4 \succ x_3 \succ x_2$
C-IFNED	$x_1 \succ x_3 \succ x_5 \succ x_4 \succ x_2$	Hurwicz	$x_3 \succ x_1 \succ x_5 \succ x_4 \succ x_2$

It can be seen that depending on the particular cases of the C-IFOWD measure, the ordering of the candidates may be different, thus leading to different decisions.

Moreover, it is possible to analyze how the different attitudinal character λ plays a role in the aggregation results, in this case, we consider different value of λ : 0, 0.1, …, 0.9, 1, which are provided by the decision makers. The collective overall values of alternatives are shown in Figure 1.

It is observed from Figure 1 that the ordering of the candidates is different, but the decisions are the same. Actually, the parameter λ, which lies in the interval $[0,1]$, can be considered as the measure of the decision maker's attitudinal character. In the extreme case, $\lambda = 1$ means that the decision make is very optimistic. On the other hand, $\lambda \to 0$ indicates that the decision maker is very conservative. Therefore, in the decision making process, if decision maker is optimistic, then we can select the parameter $\lambda \to 1$, and if decision maker is pessimistic, then we can select the parameter $\lambda \to 0$. If decision maker is neutral, then we can select the parameter $\lambda \to 0.5$.

Furthermore, we can also analyze how the different parameter value r affects the aggregation results, in this case, we consider different values of r : 0, 1, 2, …, 18, which are provided by the decision makers. The results of collective overall preference values $\tilde{r}_i$ ($i = 1,2,\cdots,5$) are shown in Figure 2.

It can be seen that depending on the particular cases of the parameter r, the ordering of the candidates may be dissimilar, thus leading to different decisions. However, it seems that x_1 is the best choice when $r \leq 5$, and x_3 sometimes is also the best one.

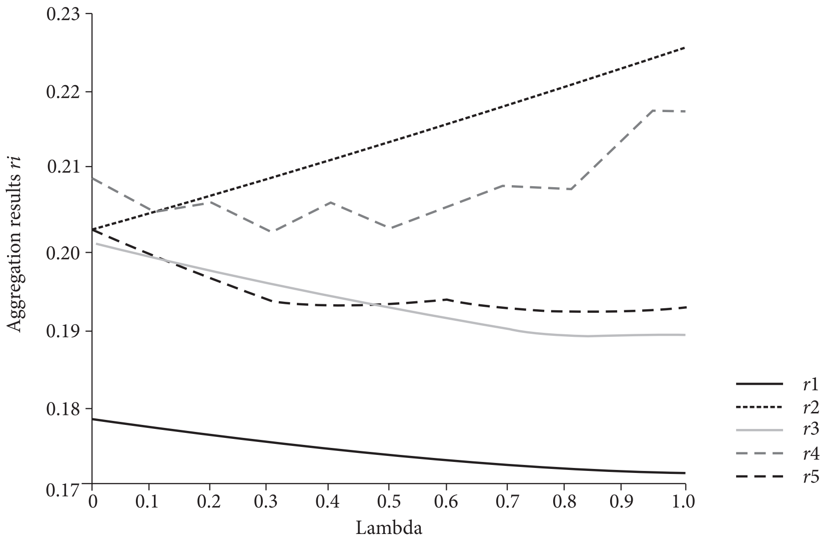

Fig. 1. Variations of the aggregation results with parameter λ

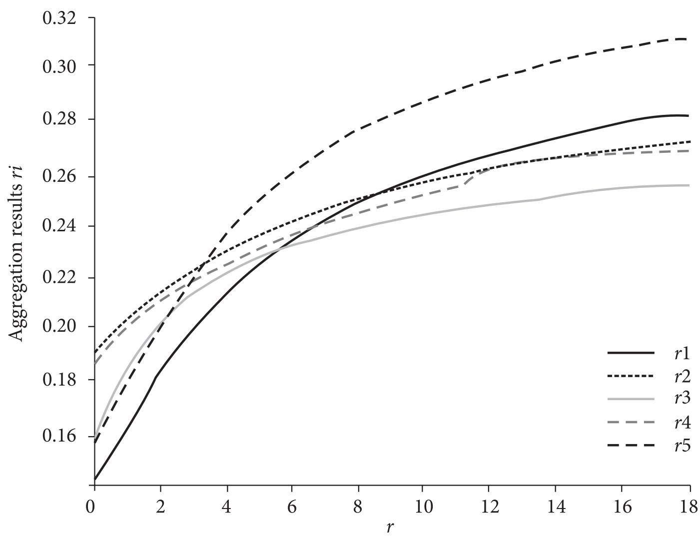

Fig. 2. Variations of the aggregation results with parameter r

Conclusions

In this paper, we have presented the C-IFOWD measure by combining the C-IVIFOWA operator with the distance measure of IVIFVs. The main advantage of the C-IFOWA operator is that it can deal with interval deviations in aggregation on IVIFVs by using a controlled parameter, which can decrease the uncertainty of argument and improve the accuracy of decision. We have further generalized the C-IFOWD measure and obtained a group of distance measures including the Quasi-C-IFOWD measure and the ∞ -C-IFOWD measure. Moreover, we have investigated some desirable properties and some families of the new distance measure. We also have proposed an application of the new approach to group decision making in an example of a human resource management problem.

In future, we expect to develop further extensions of the C-IFOWD measure to more domains by adding new characteristic or by combining it with preference relation (Gong *et al.* 2009, 2010). We will also apply it to other decision making problems, such as investment selection, product management and the strategic decision making (Gong *et al.* 2011).

Acknowledgements

The work was supported by National Natural Science Foundation of China (Nos. 71301001, 71371011, 71071002), Higher School Specialized Research Fund for the Doctoral Program (No. 20123401110001), The Scientific Research Foundation of the Returned Overseas Chinese Scholars, Anhui Provincial Natural Science Foundation (No. 1308085QG127), Provincial Natural Science Research Project of Anhui Colleges (No. KJ2012A026), Project of Anhui Province for Excellent Young Talents in Universities. The authors are thankful to the anonymous reviewers and the editor for their valuable comments and constructive suggestions with regard to this paper.

References

Atanassov, K. 1986. Intuitionistic fuzzy sets, *Fuzzy Sets and Systems* 20(1): 87–96. http://dx.doi.org/10.1016/S0165-0114(86)80034-3

Atanassov, K.; Gargov, G. 1989. Interval valued intuitionistic fuzzy sets, *Fuzzy Sets and Systems* 31(3): 343–349. http://dx.doi.org/10.1016/0165-0114(89)90205-4

Calvo, T.; Mayor, G.; Mesiar, R. 2002. *Aggregation operators: new trends and applications*. New York: Physica-Verlag. http://dx.doi.org/10.1007/978-3-7908-1787-4

Chen, H. Y.; Zhou, L. G. 2011. An approach to group decision making with interval fuzzy preference relations based on induced generalized continuous ordered weighted averaging operator, *Expert Systems with Applications* 38(10): 13432–13440. http://dx.doi.org/10.1016/j.eswa.2011.04.175

Chen, H. Y.; Zhou, L. G. 2012. A relative entropy approach to group decision making with interval reciprocal relations based on COWA operator, *Group Decision and Negotiation* 21(4): 585–599. http://dx.doi.org/10.1007/s10726-011-9228-8

Chen, S. M.; Lee, L. W.; Liu, H. C.; Yang, S. W. 2012. Multiattribute decision making based on interval-valued intuitionistic fuzzy values, *Expert Systems with Applications* 39(12): 10343–10351. http://dx.doi.org/10.1016/j.eswa.2012.01.027

Chiclana, F.; Herrera, F.; Herrera-Viedma, E. 2000. The ordered weighted geometric operator: properties and applications, in *Proc. of the 8th International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems*, 2000, Madrid, Spain, 985–991.

Gong, Z. W.; Li, L. S.; Zhou, F. X.; Yao, T. X. 2009. Goal programming approaches to obtain the priority vectors from the intuitionistic fuzzy preference relations, *Computer* & *Industrial Engineering* 57(4): 1187–1193. http://dx.doi.org/10.1016/j.cie.2009.05.007

Gong, Z. W.; Li, L. S.; Cao, J.; Zhou, F. X. 2010. On additive consistent properties of the intuitionistic fuzzy preference relation, *International Journal of Information Technology* & *Decision Making* 9(6): 1009–1025. http://dx.doi.org/10.1142/S0219622010004160

Gong, Z. W.; Li, L. S.; Forrest, J.; Zhao, Y. 2011. The optimal priority models of the intuitionistic fuzzy preference relation and their application in selecting industries with higher meteorological sensitivity, *Expert Systems with Applications* 38(4): 4394–4402. http://dx.doi.org/10.1016/j.eswa.2010.09.109

Li, D. F. 2011. The GOWA operator based approach to multiattribute decision making using intuitionistic fuzzy sets, *Mathematical and Computer Modelling* 53(5–6): 1182–1196. http://dx.doi.org/10.1016/j.mcm.2010.11.088

Liu, P. D. 2011. A weighted aggregation operator multi-attribute group decision-making method based on interval-valued trapezoidal fuzzy numbers, *Expert Systems with Applications* 38(1): 1053–1060. http://dx.doi.org/10.1016/j.eswa.2010.07.144

Liu, X. W. 2010. The orness measures for two compound quasi-arithmetic mean aggregation operators, *International Journal of Approximate Reasoning* 51(3): 305–334. http://dx.doi.org/10.1016/j.ijar.2009.10.003

Merigó, J. M. 2008. *New extensions to the OWA operator and its application in business decision making*: PhD thesis. Department of Business Administration, University of Barcelona.

Merigó, J. M. 2011. A unified model between the weighted average and the induced OWA operator, *Expert Systems with Applications* 38(9): 11560–11572. http://dx.doi.org/10.1016/j.eswa.2011.03.034

Merigó, J. M.; Casanovas, M. 2010. Decision making with distance measures and linguistic aggregation operators, *International Journal of Fuzzy Systems* 12: 190–198.

Merigó, J. M.; Casanovas, M. 2011a. Induced aggregation operators in the Euclidean distance and its application in financial decision making, *Expert Systems with Application* 38(6): 7603–7608. http://dx.doi.org/10.1016/j.eswa.2010.12.103

Merigó, J. M.; Casanovas, M. 2011b. Decision-making with distance measures and induced aggregation operators, *Computer* & *Industrial Engineering* 60(1): 66–76. http://dx.doi.org/10.1016/j.cie.2010.09.017

Merigó, J. M.; Casanovas, M. 2011c. Induced and uncertain heavy OWA operators, *Computer* & *Industrial Engineering* 60(1): 106–116. http://dx.doi.org/10.1016/j.cie.2010.10.005

Merigó, J. M.; Gil-Lafuente, A. M. 2007. On the use of the OWA operator in the Euclidean distance, *International Journal of Electrical and Systems Engineering* 2: 451–457.

Merigó, J. M.; Gil-Lafuente, A. M. 2009. The induced generalized OWA operator, *Information Sciences* 179(6): 729–741. http://dx.doi.org/10.1016/j.ins.2008.11.013

Merigó, J. M.; Gil-Lafuente, A. M. 2010. New decision making techniques and their application in the selection of financial products, *Information Sciences* 180(11): 2085–2094. http://dx.doi.org/10.1016/j.ins.2010.01.028

Merigó, J. M.; Gil-Lafuente, A. M. 2011a. Decision-making in sport management based on the OWA operator, *Expert Systems with Applications* 38(8): 10408–10413. http://dx.doi.org/10.1016/j.eswa.2011.02.104

Merigó, J. M.; Gil-Lafuente, A. M. 2011b. Fuzzy induced generalized aggregation operators and its application in multi-person decision making, *Expert Systems with Applications* 38(8): 9761–9772. http://dx.doi.org/10.1016/j.eswa.2011.02.023

Merigó, J. M.; Gil-Lafuente, A. M.; Zhou, L. G.; Chen, H. Y. 2012. Induced and linguistic generalized aggregation operators and their application in linguistic group decision making, *Group Decision and Negotiation* 21(4): 531–549. http://dx.doi.org/10.1007/s10726-010-9225-3

Mesiar, R.; Pap, E. 2008. Aggregation of infinite sequences, *Information Sciences* 178(18): 3557–3564. http://dx.doi.org/10.1016/j.ins.2008.05.020

Pereira, R. A. M.; Ribeiro, R. A. 2003. Aggregation with generalized mixture operators using weighting functions, *Fuzzy Sets and Systems* 137(1): 43–58. http://dx.doi.org/10.1016/S0165-0114(02)00431-1

Sengupta, A.; Pal, T. K. 2009. *Fuzzy preference ordering of interval numbers in decision problems*. Berlin, Heidelberg: Springer-Verlag. http://dx.doi.org/10.1007/978-3-540-89915-0

Su, Z. X.; Xia, G. P.; Chen, M. Y.; Wang, L. 2012. Induced generalized intuitionistic fuzzy OWA operator for multi-attribute group decision making, *Expert Systems with Applications* 39(2): 1902–1910. http://dx.doi.org/10.1016/j.eswa.2011.08.057

Wei, G. W. 2010a. A method for multiple attribute group decision making based on the ET-WG and ET-OWG operators with 2-tuple linguistic information, *Expert Systems with Applications* 37(12): 7895–7900. http://dx.doi.org/10.1016/j.eswa.2010.04.047

Wei, G. W. 2010b. Some induced geometric aggregation operators with intuitionistic fuzzy information and their application to group decision making, *Applied Soft Computing* 10(2): 423–431. http://dx.doi.org/10.1016/j.asoc.2009.08.009

Wei, G. W.; Zhao, X. F. 2012a. Some induced correlated aggregating operators with intuitionistic fuzzy information and their application to multiple attribute group decision making, *Expert Systems with Applications* 39(2): 2026–2034. http://dx.doi.org/10.1016/j.eswa.2011.08.031

Wei, G. W.; Zhao, X. F. 2012b. Some dependent aggregation operators with 2-tuple linguistic information and their application to multiple attribute group decision making, *Expert Systems with Applications* 39(5): 5881–5886. http://dx.doi.org/10.1016/j.eswa.2011.11.120

Wu, J.; Cao, Q. W. 2013. Same families of geometric aggregation operators with intuitionistic trapezoidal fuzzy numbers, *Applied Mathematical Modelling* 37(1–2): 318–327. http://dx.doi.org/10.1016/j.apm.2012.03.001

Wu, J.; Li, J. C.; Li, H.; Duan, W. Q. 2009. The induced continuous ordered weighted geometric operators and their application in group decision making, *Computer & Industrial Engineering* 56(4): 1545–1552. http://dx.doi.org/10.1016/j.cie.2008.09.041

Wu, J.; Cao, Q. W.; Zhang, J. L. 2010. Some properties of the induced continuous ordered weighted geometric operators in group decision making, *Computer & Industrial Engineering* 59(1): 100–106. http://dx.doi.org/10.1016/j.cie.2010.03.005

Xia, M. M.; Xu, Z. S.; Zhu, B. 2012. Some issues on intuitionistic fuzzy aggregation operators based on Archimedean t-conorm and t-norm, *Knowledge-Based Systems* 31: 78–88. http://dx.doi.org/10.1016/j.knosys.2012.02.004

Xu, Z. S. 2004. *Uncertain multiple attribute decision making: methods and applications*. Beijing: Tsinghua University Press.

Xu, Z. S. 2006a. Induced uncertain linguistic OWA operators applied to group decision making, *Information Fusion* 7(2): 231–238. http://dx.doi.org/10.1016/j.inffus.2004.06.005

Xu, Z. S. 2006b. Dependent OWA operator, *Modeling decision for artificial intelligence.* Berlin: Springer, 172–178. http://dx.doi.org/10.1007/11681960_18

Xu, Z. S. 2007a. Intuitionistic fuzzy aggregation operators, *IEEE Transactions on Fuzzy Systems* 15: 1179–1187. http://dx.doi.org/10.1109/TFUZZ.2006.890678

Xu, Z. S. 2007b. Methods for aggregating interval-valued intuitionistic fuzzy information and their application to decision making, *Control and Decision* 22: 215–219.

Xu, Z. S. 2007c. Models for multiple attribute decision making with intuitionistic fuzzy information, *International Journal of Uncertainty Fuzziness and Knowledge-Based Systems* 15(3): 285–297. http://dx.doi.org/10.1142/S0218488507004686

Xu, Z. S. 2010a. Choquet integral of weighted intuitionistic fuzzy information, *Information Sciences* 180(5): 726–736. http://dx.doi.org/10.1016/j.ins.2009.11.011

Xu, Z. S. 2010b. A method based on distance measure for interval-valued intuitionistic fuzzy group decision making, *Information Sciences* 180(1): 181–190. http://dx.doi.org/10.1016/j.ins.2009.09.005

Xu, Z. S. 2010c. A deviation-based approach to intuitionistic fuzzy multiple attribute group decision making, *Group Decision and Negotiation* 19(1): 57–76. http://dx.doi.org/10.1007/s10726-009-9164-z

Xu, Z. S. 2011. Approaches to multiple attribute group decision making based on intuitionistic fuzzy power aggregation operators, *Knowledge-Based Systems* 24(6): 749–760. http://dx.doi.org/10.1016/j.knosys.2011.01.011

Xu, Z. S. 2012. Fuzzy ordered distance measures, *Fuzzy Optimization and Decision Making* 11(1): 73–97. http://dx.doi.org/10.1007/s10700-011-9113-6

Xu, Z. S.; Cai, X. Q. 2010. Recent advances in intuitionistic fuzzy information aggregation, *Fuzzy Optimization and Decision Making* 9(4): 359–381. http://dx.doi.org/10.1007/s10700-010-9090-1

Xu, Z. S.; Chen, J. 2008a. Ordered weighted distance measure, *Journal of Systems Engineering and Electronics* 17(4): 432–445. http://dx.doi.org/10.1007/s11518-008-5084-8

Xu, Z. S.; Chen, J. 2008b. An overview of distance and similarity measures of intuitionistic fuzzy sets, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* 16(4): 529–555. http://dx.doi.org/10.1142/S0218488508005406

Xu, Z. S.; Da, Q. L. 2002a. The uncertain OWA operator, *International Journal of Intelligent Systems* 17(6): 469–483. http://dx.doi.org/10.1002/int.10038

Xu, Z. S.; Da, Q. L. 2002b. The ordered weighted geometric averaging operator, *International Journal of Intelligent Systems* 17(7): 709–716. http://dx.doi.org/10.1002/int.10045

Xu, Y. J.; Wang, H. M. 2012. The induced generalized aggregation operators for intuitionistic fuzzy sets and their application in group decision making, *Applied Soft Computing* 12(3): 1168–1179. http://dx.doi.org/10.1016/j.asoc.2011.11.003

Xu, Z. S.; Xia, M. M. 2011a. Induced generalized intuitionistic fuzzy operators, *Knowledge-Based Systems* 24(2): 197–209. http://dx.doi.org/10.1016/j.knosys.2010.04.010

Xu, Z. S.; Xia, M. M. 2011b. Distance and similarity measures for hesitant fuzzy sets, *Information Sciences* 181(11): 2128–2138. http://dx.doi.org/10.1016/j.ins.2011.01.028

Xu, Z. S.; Xia, M. M. 2011c. On distance and correlation measures of hesitant fuzzy information, *International Journal of Intelligent Systems* 26(5): 410–425. http://dx.doi.org/10.1002/int.20474

Xu, Z. S.; Yager, R. R. 2006. Some geometric aggregation operators based on intuitionistic fuzzy sets, *International Journal of General Systems* 35(4): 417–433. http://dx.doi.org/10.1080/03081070600574353

Yager, R. R. 1988. On ordered weighted averaging aggregation operators in multi-criteria decision making, *IEEE Transactions on Systems, Man and Cybernetics, B* 18: 183–190.

Yager, R. R. 1993. Families of OWA operators, *Fuzzy Sets and Systems* 59(2): 125–148. http://dx.doi.org/10.1016/0165-0114(93)90194-M

Yager, R. R. 1996. Constrained OWA aggregation, *Fuzzy Sets and Systems* 81(1): 89–101. http://dx.doi.org/10.1016/0165-0114(95)00242-1

Yager, R. R. 2003. Induced aggregation operators, *Fuzzy Sets and Systems* 137(1): 59–69. http://dx.doi.org/10.1016/S0165-0114(02)00432-3

Yager, R. R. 2004a. Generalized OWA aggregation operators, *Fuzzy Optimization and Decision Making* 3(1): 93–107. http://dx.doi.org/10.1023/B:FODM.0000013074.68765.97

Yager, R. R. 2004b. OWA aggregation over a continuous interval argument with applications to decision making, *IEEE Transactions on Systems, Man, and Cybernetics, Part B* 34: 1952–1963. http://dx.doi.org/10.1109/TSMCB.2004.831154

Yager, R. R. 2007. Centered OWA operators, *Soft Computing* 11(7): 631–639. http://dx.doi.org/10.1007/s00500-006-0125-z

Yager, R. R. 2009. Prioritized OWA aggregation, *Fuzzy Optimization and Decision Making* 8(3): 245–262. http://dx.doi.org/10.1007/s10700-009-9063-4

Yager, R. R. 2010. Norms induced from OWA operators, *IEEE Transactions on Fuzzy Systems* 18: 57–66. http://dx.doi.org/10.1109/TFUZZ.2009.2035812

Yager, R. R.; Kacprzyk, J. 1997. *The ordered weighted averaging operators: theory and applications*. Norwell, MA: Kluwer Academic Publishers. http://dx.doi.org/10.1007/978-1-4615-6123-1

Yager, R. R.; Kacprzyk, J.; Beliakov, G. 2011. *Recent developments in the ordered weighted averaging operators: theory and practice.* Berlin, Heidelberg: Springer-Verlag. http://dx.doi.org/10.1007/978-3-642-17910-5

Yager, R. R.; Xu, Z. S. 2006. The continuous ordered weighted geometric operator and its application to decision making, *Fuzzy Sets and Systems* 157(10): 1393–1402. http://dx.doi.org/10.1016/j.fss.2005.12.001

Yang, W.; Chen, Z. P. 2012. The quasi-arithmetic intuitionistic fuzzy OWA operators, *Knowledge-Based Systems* 27: 219–233. http://dx.doi.org/10.1016/j.knosys.2011.10.009

Yu, D. J.; Wu, Y. Y.; Lu, T. 2012. Interval-valued intuitionistic fuzzy prioritized operators and their application in group decision making, *Knowledge-Based Systems* 30: 57–66. http://dx.doi.org/10.1016/j.knosys.2011.11.004

Yu, X. H.; Xu, Z. S. 2013. Prioritized intuitionistic fuzzy aggregation operators, *Information Fusion* 14(1): 108–116. http://dx.doi.org/10.1016/j.inffus.2012.01.011

Yue, Z. L. 2011. Deriving decision maker's weights based on distance measure for interval-valued intuitionistic fuzzy group decision making, *Expert Systems with Applications* 38(9): 11665–11670. http://dx.doi.org/10.1016/j.eswa.2011.03.046

Zeng, S. Z.; Su, W. H. 2011. Intuitionistic fuzzy ordered weighted distance operator, *Knowledge-Based Systems* 24(8): 1224–1232. http://dx.doi.org/10.1016/j.knosys.2011.05.013

Zhang, Q. S.; Jiang, S. Y.; Jia, B. G.; Luo, S. H. 2011. Some information measures for interval-valued intuitionistic fuzzy sets, *Information Sciences* 180(24): 5130–5145. http://dx.doi.org/10.1016/j.ins.2010.08.038

Zhao, H.; Xu, Z. S.; Ni, M. F.; Liu, S. S. 2010. Generalized aggregation operators for intuitionistic fuzzy sets, *International Journal of Intelligent Systems* 25(1): 1–30. http://dx.doi.org/10.1002/int.20386

Zhou, L. G.; Chen, H. Y. 2010. Generalized ordered weighted logarithm aggregation operators and their applications to group decision making, *International Journal of Intelligent Systems* 25(7): 683–707. http://dx.doi.org/10.1002/int.20419

Zhou, L. G.; Chen, H. Y. 2011. Continuous generalized OWA operator and its application to decision making, *Fuzzy Sets and Systems* 168(1): 18–34. http://dx.doi.org/10.1016/j.fss.2010.05.009

Zhou, L. G.; Chen, H. Y. 2012. A generalization of the power aggregation operators for linguistic environment and its application in group decision making, *Knowledge-Based Systems* 26: 216–224. http://dx.doi.org/10.1016/j.knosys.2011.08.004

Zhou, L. G.; Chen, H. Y.; Liu, J. P. 2011. Generalized multiple averaging operators and their applications to group decision making, *Group Decision and Negotiation* 22(2): 331–358. http://dx.doi.org/10.1007/210726-011-9267-1

Zhou, L. G.; Chen, H. Y.; Liu, J. P. 2012a. Generalized power aggregation operators and their applications in group decision making, *Computers* & *Industrial Engineering* 62(4): 989–999. http://dx.doi.org/10.1016/j.cie.2011.12.025

Zhou, L. G.; Chen, H. Y.; Liu, J. P. 2012b. Continuous ordered weighted distance measure and its application to multiple attribute group decision making, *Group Decision and Negotiation* 22(4): 739–758. http://dx.doi.org/10.1007/s10726-012-9289-3

Zhou, L. G.; Chen, H. Y.; Merigó, J. M.; Gil-Lafuente, A. M. 2012c. Uncertain generalized aggregation operators, *Expert Systems with Applications* 39(1): 1105–1117. http://dx.doi.org/10.1016/j.eswa.2011.07.110

Ligang ZHOU received the BS, MS and PhD degrees in School of Mathematical Sciences from Anhui University, Hefei, China, in 2002, 2005 and 2013, respectively, where he is currently an Associate Professor with the School of Mathematical Sciences. He is the author or coauthor of more than 80 publications. His research interests include type-2 fuzzy sets, aggregation operators, fuzzy decision making, combination forecasting, and their applications.

Feifei JIN received the BS degree in School of Mathematics and Statistics from Hefei Normal University, Hefei, China, in 2011, the MS degree in School of Mathematical Sciences from Anhui University, Hefei, China, in 2014. He is currently a PhD student with the School of Management, Hefei University of Technology, Hefei, China. He is the author or coauthor of more than 10 publications. His research interests include aggregation operators, fuzzy decision making, combination forecasting, and their applications.

Huayou CHEN received the BS and MS degrees in School of Mathematical Sciences from Anhui University, Hefei, China, in 1991 and 1994, respectively, and the PhD degree in School of Mathematical Sciences from University of Science Technology of China, Hefei, China, in 2002. He is currently a Professor with the School of Mathematical Sciences, Anhui University, Hefei, China. He is the author or coauthor of more than 120 publications. His research interests include aggregation operators, fuzzy decision making, combination forecasting, and their applications.

Jinpei LIU received the BS and MS degrees in School of Mathematical sciences from Anhui University, Hefei, China, in 2005 and 2008, respectively, and the PhD degree in Management Science and Engineering from Tianjin University, Tianjin, China, in 2012. He is currently a Lecturer with the School of Business, Anhui University, Hefei, China. His research interests include aggregation operators, fuzzy decision making, combination forecasting, and their applications.