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ONE FUNCTIONAL OPERATOR INVERSION FORMULA

P. PLASCHINSKY

Mechanics and Mathematics Faculty, Belarusian State University

F. Skariny av. 4-420, 220050, Minsk, Belarus

E-mail: paul@mmf.bsu.unibel.by

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ABSTRACT

Some results about inversion formula of functional operator with generalized dilation are given. By means of commutative Banach algebra theory the explicit form of inversion operator is expressed. Some commutative Banach algebras with countable generator systems are constructed, their maximal ideal spaces are investigated.

1. INTRODUCTION

Examine functional operator with generalized dilation

$$(M_{a,\tau}f)(x) = \sum_{n=1}^{\infty} a(n,x)n^{\tau\nu}f(n^{\tau}x),\ 0 \neq \tau \in \mathbf{R},\ x \in (0;\infty).$$

This operator was considered in paper [3]. It were shown that the inversion formula of this operator is close to the reciprocal sequences with respect to the discrete Mellin convolution (DMC) of functional sequences with τ-degree dilation

$$(a * b)_{\tau}(n,x) = \sum_{km=n} a(k,x)b(m,k^{\tau}x),\ n \in \mathbf{N}.$$

Reciprocal sequence $a^{-1}(n,x)$ is almost everywhere on $(0;\infty)$ defined by equality

$$(a * a^{-1})_{\tau}(n,x) = (a^{-1} * a)_{\tau}(n,x) = e_1(n) = \begin{cases} 1,\ n = 1, \\ 0,\ n > 1. \end{cases}$$

In this terms the inversion formula, when all including sequences belong to special Banach space, may be expressed in form

$$M_{a,\tau}^{-1} f = M_{a^{-1},\tau} f.$$

General theory of such operators were constructed in [1]. In paper we consider more simple case of functional operator with number sequences

$$(M_{a,\tau} f)(x) = \sum_{n=1}^{\infty} a(n) n^{\tau\nu} f(n^{\tau} x)$$

in functional spaces $L_{\nu,p}$ (see [3]).

2. COMMUTATIVE BANACH ALGEBRA $\Lambda(\mathbf{N})$

Definition 2.1. Let denote by $\Lambda(\mathbf{N})$ a Banach space of absolutely summable number sequences $a(n),\ n \in \mathbf{N}$, with usual operations of summing and multiplication by number and multiplication, defined by means of DMC:

$$(a * b)(n) = \sum_{km=n} a(k) b(m). \tag{2.1}$$

In [3] were established some sufficient conditions of belonging the reciprocal sequence $a^{-1}(n)$ to l_1, i.e. sufficient conditions of invertability of $a(n)$ in $\Lambda(\mathbf{N})$. By means of maximal ideal theory we'll find inversion criterion in $\Lambda(\mathbf{N})$.

Let determine for $\Lambda(\mathbf{N})$ its maximal ideals. Set

$$N_1 = \{e_p(n),\ p - primes\},$$

were

$$e_p(n) = \begin{cases} 1, \ n = p, \\ 0, \ n \neq p, \end{cases}$$

is generator system (minimal) for commutative Banach algebra $\Lambda(\mathbf{N})$, i.e. minimal algebra, containing N_1 and multiplication unit $e_1(n)$, is $\Lambda(\mathbf{N})$.

As known [2], canonical gomomorfism by some maximal ideal M_0 is exactly defined on generator system. Let by this mapping number ζ_p corresponds to element $e_p(n)$. According to the properties of canonical gomomorfism $|\zeta_p| \leq 1$. Then to arbitrary sequence $a(n) \in \Lambda(\mathbf{N})$ corresponds number

$$a(n) \rightarrow \sum_{k=1}^{\infty} a(k) \zeta_k,$$

were ζ_k is

$$\zeta_1 = 1,\ \zeta_k = \zeta_{p_1}^{\alpha_1} \zeta_{p_2}^{\alpha_2} \cdots \zeta_{p_m}^{\alpha_m},$$

when $k = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$ — decomposition on the primes. Decomposition formula for ζ_k follows from the next property of canonical gomomorfism:

$$\zeta_p \zeta_q = \zeta_{pq},$$

because

$$(e_p * e_q)(n) = e_{pq}(n)$$

for arbitrary $p, q \in \mathbf{N}$. M_0 consists of all sequences, corresponding to zero.

In other words, M_0 corresponds to all functions, defined on set

$$B = \{z = \{z_p,\ p - primes\},\ |z_p| \leq 1\},$$

constructed by formula

$$a(z) = \sum_{k=1}^{\infty} a(k) z^k, \tag{2.2}$$

were $z^k = z_{p_1}^{\alpha_1} z_{p_2}^{\alpha_2} \cdots z_{p_m}^{\alpha_m}$, and vanished in $z_0 = (\zeta_2, \zeta_3, \ldots, \zeta_p, \ldots)$. From the theory [2] we know, that some element of commutative Banach algebra is invertable iff it does not belong to any maximal ideal. So we have obtain inversion criterion in $\Lambda(\mathbf{N})$.

Criterion 2.1. *Sequence $a(n) \in \Lambda(\mathbf{N})$ is invertable iff function $a(z)$* (2.2) *does not vanish in any point of set B.*

Corollary 2.1. Functional operator $M_{a,\tau}^{-1}$ with $a(n) \in l_1$ under the criterion 2.1 condition may be expressed as $M_{a^{-1},\tau}$, were $a^{-1}(n)$ — reciprocal to $a(n)$ with respect to DMC (2.1).

But this criterion is very difficult in use. Even in the case of finite sequence function $a(z)$ (2.2) has not very simple form. Moreover, necessary inversion condition

$$\inf_{\Re s=0} |a^*(s)| = \inf_{\Re s=0} \left| \sum_{k=1}^{\infty} a(k) k^{-s} \right| > 0, \tag{2.3}$$

obtained due to the integral Mellin transform, is very particular case of the criterion 2.1 condition. Therefore, the inversion operator $M_{a,\tau}^{-1}$ does not always expressed by means of reciprocal sequences. With the aim of keeping mentioned explicit form and simplifying criterion we extend considered algebra.

3. COMMUTATIVE BANACH ALGEBRA $\Lambda(\mathbf{Q}_{1+})$

Definition 3.1. Let denote by $\mathbf{Q}_{1+}$ set of all rationals $q \geq 1$ and $\Lambda(\mathbf{Q}_{1+})$ be a Banach space of absolutely summable number sequences $a(q)$, $q \in \mathbf{Q}_{1+}$, with usual operations of summing and multiplication by number and multiplication, defined by means of formula

$$(a * b)(q) = \sum_{rs=q} a(r)b(s),\ r, q, s \in \mathbf{Q}_{1+}.$$

This multiplication is continuous in defined space.

Obviously, algebra $\Lambda(\mathbf{N})$ is subalgebra of $\Lambda(\mathbf{Q}_{1+})$. Let investigate maximal ideals for this more extended algebra. By analogy with previous case, set

$$N_2 = \{e_r(q),\ 1 < r \in \mathbf{Q}\},$$

were

$$e_r(q) = \begin{cases} 1, & q = r, \\ 0, & q \neq r, \end{cases},\ q \in \mathbf{Q}_{1+},$$

is generator system for commutative Banach algebra $\Lambda(\mathbf{Q}_{1+})$. But canonical gomomorfism by maximal ideal M_0 it is enough to determine only on the set N_1 from section 2.

Lemma 3.1. *Maximal ideal M_0 corresponds to all functions $a(z, \alpha)$, defined on set*

$$S \times [0; \infty] = \{z = \{z_p,\ p - primes\},\ |z_p| = 1\} \times [0; \infty],$$

constructed by formula

$$a(z, \alpha) = \sum_{q \in \mathbf{Q}_{1+}} a(q) q^{-\alpha} z^q, \tag{3.1}$$

were

$$z^q = \frac{z_{p_1}^{\alpha_1} z_{p_2}^{\alpha_2} \cdots z_{p_m}^{\alpha_m}}{z_{p_{m+1}}^{\alpha_{m+1}} z_{p_{m+2}}^{\alpha_{m+2}} \cdots z_{p_{m+n}}^{\alpha_{m+n}}},$$

when

$$q = \frac{q_1}{q_2} = \frac{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}}{p_{m+1}^{\alpha_{m+1}} p_{m+2}^{\alpha_{m+2}} \cdots p_{m+n}^{\alpha_{m+n}}},\ q_1 > q_2$$

— fraction decomposition on different primes, and vanished in some fixed point of $S \times [0; \infty]$.

Proof. Actually, we must prove that for arbitrary r number ζ_r, corresponding to element $e_r(q)$, has absolute, equal to $r^{-\alpha}$. Let consider 3 different cases.

1. If any of $\zeta_r = 0$, then for all $s > 1$ we obtain $\zeta_s = 0$. Really, for all $r > s$

$$\zeta_s = \zeta_r \zeta_{s/r} = 0$$

by the property, mentioned in section 2. If $1 < s < r$, then there is such natural n_s, that $s^{n_s} > r$ and

$$\zeta_{s^{n_s}} = \zeta_s^{n_s} = 0 \Rightarrow \zeta_s = 0.$$

So we may define this case as $\alpha = +\infty$.

2. If any of $|\zeta_r| = 1$, then for all s we obtain $|\zeta_s| = 1$. This follows from the next inequality

$$|\zeta_r| \le |\zeta_s| \text{ for any } s < r,$$

obtained from $|\zeta_{r/s}| \le 1$.

In fact, for all $r > s > 1$

$$1 \ge |\zeta_s| \ge |\zeta_r| = 1 \ \Rightarrow \ |\zeta_s| = 1.$$

As $r^n \to \infty$, when $n \to \infty$, statement is proved for arbitrary s. So we may define this case as $\alpha = 0$.

3. All $|\zeta_r| \ne 0, 1$. Suppose $|\zeta_r| = r^{-\alpha}$, but there is $s > r$, for which $|\zeta_s| = s^{-\beta}$. It is easy to prove that $|\zeta_r| > |\zeta_s|$ for any $s > r$.

If $\alpha > \beta$, there are naturals n and m, such that $r^n < s^m$, but $|\zeta_{r^n}| < |\zeta_{s^m}|$. This follows from existence of rational $\frac{n}{m}$, satisfying inequality

$$\frac{\beta}{\alpha} \cdot \frac{\ln s}{\ln r} < \frac{n}{m} < \frac{\ln s}{\ln r},$$

which contradicts with properties of canonical gomomorfism.

Case $\alpha < \beta$ is wholly analogous.

Decomposition formula for z^q is proved as in section 2. ■

Immediately from this lemma we obtain inversion criterion in $\Lambda(\mathbf{Q}_{1+})$.

Criterion 3.1. *Sequence $a(q) \in \Lambda(\mathbf{Q}_{1+})$ is invertable iff function $a(z, \alpha)$ (3.1) does not vanish in any point of set $S \times [0; \infty]$.*

The following lemma allows us simplify criterion 3.1 condition.

Lemma 3.2. *Sequence $a(q) \in \Lambda(\mathbf{Q}_{1+})$ is invertable iff*

$$\inf_{\Re s \ge 0} \left| \sum_{q \in \mathbf{Q}_{1+}} a(q) q^{-s} \right| > 0.$$

Proof. Statement of the lemma follows from the fact that maximal ideals, corresponding to functions r^{-s}, $r \in \mathbf{Q}_{1+}$, are dense in $\Lambda(\mathbf{Q}_{1+})$ maximal ideal space (see proof of theorem 2, [2], p. 189). ■

But our functional operator $M_{a,\tau}$ is defined on sequences from $\Lambda(\mathbf{N})$. Therefore let transfer the results obtained on mentioned subalgebra.

Theorem 3.1. *Banach algebra* $\Lambda(\mathbf{N})$ *is filled subalgebra of* $\Lambda(\mathbf{Q}_{1+})$, *i.e. sequence* $a(n)$ *is invertable in* $\Lambda(\mathbf{N})$ *iff it is invertable in* $\Lambda(\mathbf{Q}_{1+})$.

Proof.
Necessity. It is evident, that sequence $a^{-1}(n)$ from $\Lambda(\mathbf{N})$ is reciprocal sequence in $\Lambda(\mathbf{Q}_{1+})$ too.
Sufficiency. Let show that $a^{-1}(q)$ belongs to $\Lambda(\mathbf{N})$, i.e. $a^{-1}(q) = 0$ for all $q \neq n \in \mathbf{N}$. Obviously, $a(1) \neq 0$ (necessary inversion condition).
Let $q \in (1;2)$. Then by the formula from definition 3.1 and by the reciprocal definition

$$(a * a^{-1})(q) = a(1)a^{-1}(q) = 0 \;\Rightarrow\; a^{-1}(q) = 0,\; q \in (1;2).$$

Suppose

$$a^{-1}(q) = 0,\; q \in \bigsqcup_{i=1}^{m-1} (i; i+1).$$

Let $q \in (m; m+1)$. Then $q/k < m$ and $q/k \notin \mathbf{N}$ for $k = \overline{2, m}$. From equality

$$(a * a^{-1})(q) = a(1)a^{-1}(q) + a(2)a^{-1}(q/2) + \ldots + a(m)a^{-1}(q/m) = 0$$

follows $a^{-1}(q) = 0$. So we have complete the proof of the theorem. ■

Corollary 3.1. Sequence $a(n) \in \Lambda(\mathbf{N})$ is invertable iff

$$\inf_{\Re s \geq 0} |a^*(s)| = \inf_{\Re s \geq 0} \left| \sum_{k=1}^{\infty} a(k) k^{-s} \right| > 0. \tag{3.2}$$

In the same time operator $M_{a,\tau}^{-1}$ may be expressed as $M_{a^{-1},\tau}$, were $a^{-1}(n)$ — reciprocal sequence to $a(n)$ with respect to DMC (2.1).

Notify that corollary 3.1 inversion condition (3.2) is more strong then necessary inversion condition (2.3). That is why let continue to extend Banach algebra $\Lambda(\mathbf{N})$.

4. COMMUTATIVE BANACH ALGEBRA $\Lambda(\mathbf{Q}_+)$

Definition 4.1. Let denote by $\mathbf{Q}_+$ set of all positive rationals and $\Lambda(\mathbf{Q}_+)$ be a Banach space of absolutely summable number sequences $a(q)$, $q \in \mathbf{Q}_+$, with usual operations of summing and multiplication by number and multiplication, defined by means of formula

$$(a * b)(q) = \sum_{r \in \mathbf{Q}_+} a(r) b(q/r),\; q \in \mathbf{Q}_+.$$

The multiplication thus defined is continuous in $\Lambda(\mathbf{Q}_+)$.

It is evident that

$$\Lambda(\mathbf{N}) \subset \Lambda(\mathbf{Q}_{1+}) \subset \Lambda(\mathbf{Q}_+).$$

Let investigate $\Lambda(\mathbf{Q}_+)$ maximal ideal space. As in section 3 set

$$N_3 = \{e_r(q),\ r \in \mathbf{Q}_+,\ r \neq 1\},$$

were

$$e_r(q) = \begin{cases} 1, & q = r, \\ 0, & q \neq r, \end{cases},\ q \in \mathbf{Q}_+,$$

is generator system for commutative Banach algebra $\Lambda(\mathbf{Q}_+)$. But canonical gomomorfism by maximal ideal M_0 enough to determine only on the set N_1 from section 2.

Lemma 4.1. *Maximal ideal M_0 corresponds to all functions $a(z)$, defined on set*

$$S = \{z = \{z_p,\ p - primes\},\ |z_p| = 1\},$$

constructed by formula

$$a(z) = \sum_{q \in \mathbf{Q}_+} a(q) z^q, \tag{4.1}$$

were

$$z^q = \frac{z_{p_1}^{\alpha_1} z_{p_2}^{\alpha_2} \cdots z_{p_m}^{\alpha_m}}{z_{p_{m+1}}^{\alpha_{m+1}} z_{p_{m+2}}^{\alpha_{m+2}} \cdots z_{p_{m+n}}^{\alpha_{m+n}}},$$

when

$$q = \frac{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}}{p_{m+1}^{\alpha_{m+1}} p_{m+2}^{\alpha_{m+2}} \cdots p_{m+n}^{\alpha_{m+n}}}$$

— fraction decomposition on different primes, and vanished in some fixed point of S.

Proof. From $(e_r * e_{1/r})(q) = e_1(q)$ follows $|\zeta_{1/r}| = 1/|\zeta_r|$. As known [2] $|\zeta_r| \leq 1$. So $|\zeta_r| = 1$ for any positive rational r.

Decomposition formula for z^q is proved as in lemma 3.1. ■

So we have obtain inversion criterion in $\Lambda(\mathbf{Q}_+)$.

Criterion 4.1. *Sequence $a(q) \in \Lambda(\mathbf{Q}_+)$ is invertable iff function $a(z)$ (4.1) does not vanish in any point of set S.*

Lemma 4.2. *Criterion* (4.1) *condition may be rewrite as*

$$\inf_{\Re s = 0} \left| \sum_{q \in \mathbf{Q}_+} a(q) q^{-s} \right| > 0.$$

Proof of the lemma is as in lemma 3.2.

Corollary 4.1. If

$$\inf_{\Re s=0} |a^*(s)| = \inf_{\Re s=0} \left| \sum_{k=1}^{\infty} a(k)k^{-s} \right| > 0,$$

then operator $M_{a,\tau}^{-1}$ may be expressed as $M_{a^{-1},\tau}$, were $a^{-1}(q)$ — reciprocal sequence to $a(n)$ in $\Lambda(\mathbf{Q}_+)$.

5. FINAL CRITERION

The results thus obtained allow us to unite all corollaries in one inversion criterion for functional operator $M_{a,\tau}$.

Theorem 5.1. *Operator $M_{a,\tau}$ with $a(n) \in l_1$ has inversion in $L_{\nu,p}$ iff*

$$\inf_{\Re s=0} |a^*(s)| = \inf_{\Re s=0} \left| \sum_{k=1}^{\infty} a(k)k^{-s} \right| > 0,$$

and inversion operator $M_{a,\tau}^{-1}$ may be expressed as $M_{a^{-1},\tau}$, were $a^{-1}(q)$ — reciprocal sequence to $a(n)$ in $\Lambda(\mathbf{Q}_+)$.

Moreover, if

$$\inf_{\Re s\geq 0} |a^*(s)| = \inf_{\Re s\geq 0} \left| \sum_{k=1}^{\infty} a(k)k^{-s} \right| > 0,$$

then in the mentioned explicit form $a^{-1}(n)$ — reciprocal sequence to $a(n)$ in $\Lambda(\mathbf{N})$.

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VIENO FUNKCINIO OPERĄTORIAUS ATVIRKŠTINIO OPERATORIAUS FORMULĖ

P. Plachinsky

Straipsnyje nagrinėjama vieno apibendrinto operatoriaus apvertimo problema. Naudojantis komutatyviąja Banacho algebros teorija, sudaryta atvirkštinio operatoriaus išreikštinė skaičiavimo formulė. Sukonstruotos kelios komutatyvinės Banacho algebros, kurių generuojančios sistemos yra suskaičiuojamos. Pagrindinėje teoremoje pateiktos būtinos ir pakankamos sąlygos, kad funkcinis operatorius turėtų atvirkštinį.