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FEDORENKO FINITE SUPERELEMENT METHOD AS SPECIAL GALERKIN APPROXIMATION [1]

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ABSTRACT

In this work we introduce variational equation which natural Petrov-Galerkin approximation leads to Fedorenko Finite Superelement Method (FSEM). FSEM is considered as Petrov-Galerkin approximation of the certain problem for traces of boundary-value problem solution at the boundaries of some subdomains (superelements). Iterative methods of solution of the same problem are well known domain decomposition methods. Some numerical results are presented.

1. INTRODUCTION

This paper considers some results on error analysis and applications of Fedorenko Finite Superelement Method [2; 10; 11; 12]. Error analysis for one variant of this method is presented in [7; 8]. Using the variational equation constructed below and suitable Petrov-Galerkin approximation one can consider a whole class of FSEM methods, not only the method introduced in original works of Fedorenko and his colleagues. In this paper we consider FSEM for Poisson equation in multiply-connected two-dimensional domain, but all results can be generalized over equations with arbitrary positively defined divergent second order elliptic operators.

FSEM belongs to the class of methods which reduce the initial boundary value problem in whole domain to a number of boundary value problems in

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subdomains. Such methods (for example, domain decomposition methods) are widely used and studied currently. They are naturally parallelizable and can be effectively realized on computers with parallel or multiprocessor architecture. The theoretical background for this methods is Poincare-Steklov operator's theory, which was introduced firstly in works of V. Agoshkov and V. Lebedev as effective tool for analysis of domain-decomposition methods (see review in [4]). This work also widely uses this formalism.

2. PROBLEM FORMULATION

We shall use conventional notations for functional spaces of smooth functions and functions with distributional derivatives [1; 4; 6].

Let us consider the following Dirichlet boundary value problem for Poisson equation in domain $\Omega \subset \mathbb{R}^2$. Find $u \in W_2^1(\Omega, -\Delta)$ such that

$$-\Delta u = f, \quad x \in \Omega, \tag{2.1}$$

$$u|_{\partial\Omega} = g, \tag{2.2}$$

where $f \in L_2(\Omega), \quad g \in W_2^{1/2}(\partial\Omega)$,

$$W_2^1(\Omega, -\Delta) = \left\{u : u \in W_2^1(\Omega),\ -\Delta u \in L_2(\Omega)\right\}.$$

Here we suppose that $\Omega \subset \mathbb{R}^2$ is an open multiply-connected domain which can be generated from open connected domain $\Omega_0 \subset \mathbb{R}^2$ by elimination of some disjoint circles $\{\overline{S}_i\}$ ("wells") $\underset{i}{\cup}\overline{S}_i \subset \Omega_0$, $\Omega = \Omega_0 \setminus \underset{i=1}{\cup}\overline{S}_i$. We denote by $\Gamma_0 = \partial\overline{\Omega}_0$, $\Gamma_i = \partial\overline{S}_i$ the exterior and inner boundary of Ω, respectively. Therefore, we have $\overline{\Omega} = \Omega \cup \Gamma$, $\Gamma \equiv \partial\Omega = \Gamma_0 \cup \left(\underset{i}{\cup}\partial\overline{S}_i\right)$. Boundary condition (2.2) defines the trace of the solution at the whole disconnected boundary of Ω.

3. WEAK FORMULATIONS

3.1. Green's formula

Green's formula is the main tool for construction and analysis of weak formulations of boundary-value problems. We will use the following variant of Green's formula in $W_2^1(\Omega)$, which follows from abstract Green's formula [6, p.188]. It states that there exists the unequally defined operator

$$\delta : W_2^1(\Omega, -\Delta) \to W_2^{-1/2}(\partial\Omega),$$

such that $a(u,v) = (-\Delta u, v)_\Omega + \langle \delta u, \gamma v\rangle_{\partial\Omega}$, $\forall u \in W_2^1(\Omega, -\Delta)$, $v \in W_2^1(\Omega)$, $W_2^1(\Omega, -\Delta) = \left\{u \in W_2^1(\Omega) : -\Delta u \in L_2(\Omega)\right\}$, $a(u,v) = \int\limits_\Omega \nabla u \cdot \nabla v \, d\Omega$. Here

$\gamma : W_2^1(\Omega) \to W_2^{1/2}(\partial\Omega)$ is a trace operator. If $u \in W_2^2(\Omega)$ and boundary $\partial\Omega$ is sufficiently smooth, then

$$\delta u = \frac{\partial u}{\partial \vec{n}}, \quad \langle \delta u, \gamma v \rangle_{\partial\Omega} = \int\limits_{\partial\Omega} \frac{\partial u}{\partial \vec{n}} \, v \, d\gamma.$$

3.2. Classical weak equation

Classical weak formulation (2.1)–(2.2) is given by [6]. Find $u \in W_2^1(\Omega)$ such that

$$u|_{\partial\Omega} = g, \quad g \in W_2^{1/2}(\partial\Omega), \tag{3.1}$$

$$a(u,v) = (f,v), \quad \forall v \in \overset{\circ}{W}{}_2^1(\Omega); \quad f \in L_2(\Omega). \tag{3.2}$$

Problems (2.1)-(2.2) and (3.1)-(3.2) are equivalent [6, chapter 6]. Formulation (3.1) – (3.2) is also valid if $f \in W_2^{-1}(\Omega)$, but in this case it is not equivalent to (2.1) – (2.2).

4. SUPERELEMENTS

Suppose that domain $\overline{\Omega}_0$ is decomposed in K subdomains $\{\overline{\Omega}_{0,k}\}_{k=1}^K : \overline{\Omega}_0 = \bigcup\limits_{k=1}^{K} \overline{\Omega}_{0,k}, \quad \overline{\Omega}_{0,k} = \Omega_{0,k} \cup \Gamma_{0,k}, \quad \Gamma_{0,k} = \partial\overline{\Omega}_{0,k}$, in such a way, that every $\overline{S}_i$ is an inner subdomain of $\Omega_{0,k}$ and every $\Omega_{0,k}$ contains no more then one "well" S_i and decomposition is regular. Let $\Omega_k = \Omega \cap \Omega_{0,k}$. Then $\overline{\Omega} = \bigcup\limits_{k=1}^{K} \overline{\Omega}_k$. We can represent boundary $\partial\Omega_k$ of domain Ω_k as $\partial\Omega_k = \gamma_{0,k} \cup \gamma_k$, $\gamma_{0,k} = \partial\Omega \cap \partial\Omega_k$, $\gamma_k = \partial\Omega_k \setminus \gamma_{0,k}$. We shall say that Ω_k are superelements. Since all subdomains are polygons, they have outer normals almost everywhere.

5. VARIATIONAL EQUATION IN THE TRACE SPACES

5.1. Poincare-Steklov operator

Let the operator $G : W_2^{1/2}(\partial\Omega) \to W_2^1(\Omega)$ is such that

$$G\varphi = u,$$

$$u|_{\partial\Omega} = \varphi, \quad \varphi \in W_2^{1/2}(\partial\Omega), \tag{5.1}$$

$$a(u,v) = 0, \quad \forall v \in \overset{\circ}{W}{}_2^1(\Omega). \tag{5.2}$$

It is well-known that G is a linear continuous operator, e.g. $\forall c_1, c_2 \in \mathbb{R}$

$$G(c_1\varphi_1 + c_2\varphi_2) = c_1 G(\varphi_1) + c_2 G(\varphi_2); \quad \|G\varphi\|_{W_2^1(\Omega)} \le C \|\varphi\|_{W_2^{1/2}(\partial\Omega)},$$

where $C > 0$ is independent from φ.

We define the following Poincare-Steklov operator:
$P : W_2^{1/2}(\partial\Omega) \to W_2^{-1/2}(\partial\Omega)$, $P\varphi = \delta G\varphi$. If solution $u = G\varphi \in W_2^2(\Omega)$, then $P\varphi = \partial G\varphi/\partial\vec{n}$. It follows from Green's formula that $\forall\varphi_1, \varphi_2 \in W_2^{1/2}(\partial\Omega)$: $\langle P\varphi_1, \varphi_2\rangle = \langle\varphi_1, P\varphi_2\rangle$. We will denote operators G and P corresponding to domain Ω_k as G_k and P_k.

5.2. Variational equation

Let's consider the following Hilbert space $\tilde{H}$:

$$\tilde{H} = \prod_{k=1}^{K} L_2(\partial\Omega_k), \quad \langle\mu, \nu\rangle_{\tilde{H}} = \sum_{k=1}^{K}\langle\mu_k, \nu_k\rangle_{\partial\Omega_k}, \quad \|\mu\|_{\tilde{H}} = \sqrt{\langle\mu, \mu\rangle_{\tilde{H}}}.$$

Let H be a subspace of $\tilde{H}$:

$$H = \Big\{\mu \in \tilde{H} : \quad \forall i, j \in \{1, \dots, K\} \text{ such that } \gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j \neq \emptyset,$$
$$\mu_i|_{\gamma_{ij}} = \mu_j|_{\gamma_{ij}} \text{ almost everywhere}\Big\}.$$

Also we consider the Hilbert space

$$\tilde{X} = \prod_{k=1}^{K} W_2^{1/2}(\partial\Omega_k), \quad \|\mu\|_{\tilde{X}} = \sum_{k=1}^{K} \|\mu_k\|_{W_2^{1/2}(\partial\Omega_k)}$$

and it's subspaces

$$X = \Big\{\mu \in \tilde{X} : \exists v \in W_2^1(\Omega),\, \mu_k = v|_{\partial\Omega_k}\Big\},$$
$$X_0 = \Big\{\mu \in \tilde{X} : \exists v \in \overset{\circ}{W}{}_2^1(\Omega),\, \mu_k = v|_{\partial\Omega_k}\Big\}.$$

Let $\tilde{X}' = \prod\limits_{k=1}^{K} W_2^{-1/2}(\partial\Omega_k)$ be a dual space to $\tilde{X}$. Since $W_2^{1/2}(\partial\Omega_k) \subset L_2(\partial\Omega_k) \subset W_2^{-1/2}(\partial\Omega_k)$, we have $\tilde{X} \subset H \subset \tilde{X}'$. Notice that $X' = \tilde{X}'$. In the spaces $W_2^{-1/2}(\partial\Omega_k)$ and X' we use standard dual norms. Now we consider Poincare-Steklov operators P_k and a bilinear form

$$b(\mu, \nu) = \sum_{k=1}^{K} \langle P_k\mu_k, \nu_k\rangle_{\partial\Omega_k},$$

defined for all $\mu, \nu \in \tilde{X}$. The following lemma is valid.

Lemma 5.1. *Form* $b(\cdot,\cdot)$ *is a continuous symmetric positively defined bilinear form in* $X_0 \times X_0$*, e.g.* $\forall \mu, \nu \in X_0 : \quad |b(\mu,\nu)| \le c_2 \|\mu\|_{X_0} \|\nu\|_{X_0}$; $\forall \mu \in X_0 :$ $c_1 \|\mu\|^2_{X_0} \le b(\mu,\mu)$.

Now we can formulate the following problem: find $\varphi \in X_0$ such that

$$b(\varphi,\psi) = F(\psi), \quad \forall \psi \in X_0, \tag{5.3}$$

where $F(\psi)$ is a continuous linear functional on X, $F \in X'$.

It follows from Lax-Milgramm lemma that this problem has the unique solution. Finally, we can prove the following theorem.

Theorem 5.1. *Let* $F(\psi) = -b(\tilde{\varphi},\psi)$, $\psi \in X$; $\tilde{\varphi} = \{\tilde{\varphi}_k\}_{k=1}^K = \{v|_{\partial\Omega_k}\}_{k=1}^K \in X$, $F \in X'$, *where* $v = \{v_k\}_{k=1}^K$ *is such a function that*

$$-\Delta v_k = f \quad in \quad \Omega_k, \quad v_k|_{\partial\Omega_k} = w|_{\partial\Omega_k},\ k = \overline{1,K},$$

and $w \in W_2^1(\Omega)$, $w|_{\partial\Omega} = g$ *is any function satisfying* (2.2). *Then solution* $u = \{u_k\}_{k=1}^K \in W_2^1(\Omega)$ *of the problem* (2.1) – (2.2) *in superelement* Ω_k *is given by* $u_k = G_k\varphi_k + v_k$, $k = \overline{1,K}$.

Suppose that in (5.3) the functional $F \in X' = \tilde{X}'$ belongs to $\tilde{H} = \tilde{H}' \subset \tilde{X}'$. Then $F(\psi) = \langle F, \psi\rangle = \sum\limits_{k=1}^{K} \int\limits_{\partial\Omega_k} F_k\, \psi_k dl$ and we can reformulate the problem (5.3) in the following way. Find $\varphi \in X_0$ such that

$$b(\varphi,\psi) = F(\psi), \quad \forall \psi \in H_0, \tag{5.4}$$

where H_0 is subspace of H, which consists of functions, vanishing on $\gamma_{0,k} = \partial\Omega_k \cap \partial\Omega$. In this case the bilinear form $b(\cdot,\cdot)$ is defined in $X_0 \times H_0$ and it is not symmetric.

6. APPROXIMATIONS

Finite Superelement Method is a Petrov-Galerkin approximation of (5.4). Petrov-Galerkin approximations of abstract and boundary-value problems are considered in [3; 5].

For the approximation of the problem (5.4) one should choose the finite dimensional subspace $X_{0,h}$ which is a span of function system $\{\varphi_h^{(i)}\}_{i=1}^N \subset X_0$, and the finite dimensional subspace $H_{0,h}$, which is a span of function system $\{\psi_h^{(i)}\}_{i=1}^N \subset H_0$. We find the approximate solution $\varphi_h \in X_{0,h}$ of the problem (5.4) as the following linear combination:

$$\varphi_h = \sum_{i=1}^{N} a_i \varphi_h^{(i)} \tag{6.1}$$

satisfying the conditions:

$$b(\varphi_h, \psi_h) = F(\psi_h), \quad \forall \psi_h \in H_{0,h}. \tag{6.2}$$

Choosing different subspaces $X_{0,h}$ and $H_{0,h}$ one can get different variants of FSEM. In all cases the algorithm implementation is given by:

1. For every function $\varphi_h^{(i)}$ compute and store the following functions:

$$u_h^{(i)} = G\varphi_h^{(i)} = \{G_k \varphi_{h,k}^{(i)}\}_{k=1}^{K},$$

$$\Pi_h^{(i)} = P\varphi_h^{(i)} = \{P_k \varphi_{h,k}^{(i)}\}_{k=1}^{K} = \{\partial u_{h,k}^{(i)} / \partial n_k\}_{k=1}^{K}.$$

   To compute $u_h^{(i)}$ one should solve in each Ω_k the boundary value problem with the boundary condition, defined by function $\varphi_h^{(i)}$.
2. After solving problem (6.2), we get approximate solution φ_h of the problem (5.4). Then the approximate solution of the boundary value problem (3.1)-(3.2) is given by $u_h = \sum\limits_{i=1}^{N} a_i u_h^{(i)}$.

We consider two variants of $X_{0,h}$ and $H_{0,h}$.

6.1. Fedorenko FSEM

On $\bigcup\limits_{k=1}^{K} \partial\Omega_k$ we consider the set of nodes M_h consisting of vertices of polygons Ω_k. For every node $M_i \in M_h$ we define a basis function $\varphi_h^{(i)} \in X_{0,h}$ in the following way: $\varphi_h^{(i)}(M_j) = \delta_{ij}$. At exterior bounds of superelements $\varphi_h^{(i)}$ is a linear function, which vanishes at inner bounds ("wells"). Function $\psi_h^{(i)}$ is defined as characteristic function of ω_i. If we choose $X_{0,h}$ and $H_{0,h}$ as shown above, we get the first order Fedorenko FSEM.

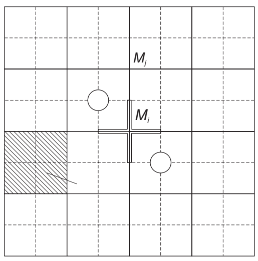

Figure 1. Domain and superelements.

6.2. Bubnov-Galerkin FSEM

Bubnov-Galerkin FSEM corresponds to the case when $\varphi_h^{(i)} = \psi_h^{(i)}$ (systems of test and trial functions are equal). Basis functions $\varphi_h^{(i)}$ are defined as presented above.

7. BUBNOV-GALERKIN FSEM ERROR ANALYSIS

It is well known that the error analysis of Bubnov-Galerkin approximations leads to the analysis of the interpolation errors of the solution φ by finite dimensional subspace $X_{0,h}$. According to Sea lemma [1, p.109] we obtain:

$$\|\varphi - \varphi_h\|_X \le M \inf_{\psi_h \in V_h} \|\varphi - \psi_h\|_X \le \|\varphi - \tilde{\varphi}\|_X \,, \quad M = c_2/c_1. \tag{7.1}$$

Here φ_h is the solution of (6.2) with $\psi_h^{(i)} = \varphi_h^{(i)}$ and $\tilde{\varphi} \in X_{0,h}$ is the interpolating element of the exact solution φ in $X_{0,h}$.

Now we can estimate $\|\varphi - \tilde{\varphi}\|_X$ in the case of polynomial interpolation. Additionally we suppose that $u \in W_2^1(\Omega) \cap C^2(\Omega)$. In this case i-th basis function $\varphi_h^{(i)} \in X_{0,h}$ is defined as interpolating polynomial of arbitrary function from $\varphi \in X_0$, which is equal to one in i-th node and is equal to zero in all other nodes.

Let $I = [a, b] \subset \mathbb{R}$. We consider $W_2^{1/2}(I)$ space on I with norm

$$\|\varphi\|^2_{W_2^{1/2}(I)} = \|\varphi\|^2_{L_2(I)} + \int\limits_I \int\limits_I \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^2}\, dxdy; \; \|\varphi\|^2_{L_2(I)} = \int\limits_I \varphi^2(x)\, dx.$$

Let $\varphi \in C^1(I) \cap W_2^{1/2}(I)$. It follows from (7.1) that

$$\|\varphi\|^2_{W_2^{1/2}(I)} \le \|\varphi\|^2_{L_2(I)} + \alpha^2 |I|^2, \quad \alpha = \max_{\xi \in I} |\varphi'(\xi)|, \quad |I| = |b-a|.$$

Suppose that $\varphi \in C_0^1(I)$. Then we obtain

$$\|\varphi\|^2_{W_2^{1/2}(I)} \le \frac{1}{3}\alpha^2 |I|^3 + \alpha^2 |I|^2 = \alpha^2 |I|^2 (\frac{1}{3}|I| + 1). \tag{7.2}$$

Now we consider the domain $\Omega \subset \mathbb{R}^2$ and its regular decomposition in K polygons Ω_k, $k = \overline{1, K}$. Let $\Omega_0 = \mathbb{R}^2 \setminus \Omega$. Let $\gamma_{i,j}$ be a common edge of subdomains Ω_i and Ω_j, $i, j = \overline{0, K}$, $i < j$. We denote the set of all such pairs by J. Functional space X can be defined as follows:

$$X = \left\{ \mu = \{\mu_k\}_{k=1}^K \in \tilde{X} : \quad \forall\, (i,j) \in J \quad \mu_i|_{\gamma_{i,j}} = \mu_j|_{\gamma_{i,j}} \right\}.$$

We shall denote $\|\mu_k\|^2_{W_2^{1/2}(\partial\Omega_k)} = \sum\limits_{i,j} \|\mu_k\|^2_{W_2^{1/2}(\gamma_{i,j})}$, where summation is done through all pairs $(i,j) \in J$, such that $\gamma_{i,j} \in \partial\Omega_k$. Then we obtain

$$\|\mu\|^2_X = \sum_{k=1}^K \|\mu_k\|^2_{W_2^{1/2}(\partial\Omega_k)} = 2 \sum_{i,j \in J} \|\mu_k\|^2_{W_2^{1/2}(\gamma_{i,j})}\,. \tag{7.3}$$

At the $\bigcup\limits_{k=1}^{K} \partial\Omega_k$ we consider the set of nodes M_h. We assume that every edge $\gamma_{i,j}$ has the same number of nodes $L+1 \geq 2$ and two of this nodes always correspond to the vertices of $\gamma_{i,j}$. Interpolating element of $\varphi \in X$ at every edge $\gamma_{i,j}$ is defined as ordinary interpolating polynom of $\varphi|_{\gamma_{i,j}}$ over $L+1$ nodes at $\gamma_{i,j}$. Here we assume that $\varphi_k \in W_2^{1/2}(\partial\Omega_k)$ is continuous function and $\varphi|_{\gamma_{i,j}} \in C^1(\gamma_{i,j})$. Combining (7.3) and (7.2), we reduce the estimates for $\|\varphi - \tilde{\varphi}\|_X$ to the error estimates of ordinary polynomial interpolation at $\gamma_{i,j}$.

Let $M_{i,j}$ be a set of nodes which belong to edge $\gamma_{i,j}$. At edge $\gamma_{i,j}$ we consider some coordinate system Os with the origin at one of the vertices of $\gamma_{i,j}$. Let $\{0 = s_0,\, s_1, \ldots, s_{L-1},\, s_L = |\gamma_{i,j}|\}$ be coordinates of nodes from $M_{i,j}$. By $\varphi_{i,j}$ and $\tilde{\varphi}_{i,j}$ we denote the restrictions of φ and $\tilde{\varphi}$ on $\gamma_{i,j}$. Then interpolation errors obey the following inequalities [9]

$$\max_{s\in\gamma_{i,j}} |\varphi_{i,j} - \tilde{\varphi}_{i,j}| \leq \frac{\alpha_{L+1,i,j}}{(L+1)!} |\gamma_{i,j}|^{L+1},$$

$$\max_{s\in\gamma_{i,j}} |\varphi'_{i,j} - \tilde{\varphi}'_{i,j}| \leq \frac{\alpha_{L+1,i,j}}{L!} |\gamma_{i,j}|^{L}; \quad \alpha_{L+1,i,j} = \max_{s\in\gamma_{i,j}} \left|\varphi^{(L+1)}(s)\right|. \tag{7.4}$$

Finally, combining (7.2), (7.3) and (7.4) we obtain

$$\|\varphi - \tilde{\varphi}\|_X^2 \leq 2 \sum_{i,j\in J} \frac{\alpha^2_{L+1,i,j}}{(L!)^2} |\gamma_{i,j}|^{2(L+1)} (\frac{1}{3}|\gamma_{i,j}| + 1). \tag{7.5}$$

Here $\alpha_{L+1,i,j}$ depend on the domain Ω decomposition.

From the well known properties of polynomial interpolation it follows that most interesting cases are $L=1$ and $L=2$. It also follows that the traces of the solution at the boundaries of superelements should be sufficiently smooth functions. But inside superelements the solution can have singularities or large gradients. Suitable way of interpolation also depends on superelement geometry, type and location of singularities. In general case at every edge $\gamma_{i,j}$ one can use different kind of interpolation. Taking into account the continuity of operators G_k we obtain that the convergence of φ_h to φ in X leads to the convergence of u_h to u in $W_2^1(\Omega)$.

It is interesting to study the dependence of error estimates on distance between "well" and superelement boundary. We consider the boundary value problem for the Laplace equation in the square domain with one "well" in the center of the square. We assume that edges of the square have length l and are parallel to the coordinate axes, the "well" has radius r_c and it is situated at the origin of coordinates. We will use the following function as an exact solution of the model problem:

$$u(r,\varphi) = u_c + \ln\left(\frac{r_c}{r}\right), \tag{7.6}$$

where (r,φ) are polar coordinates in Oxy. This function satisfies the Laplace equation in Ω and it is equal to u_c at the boundary of the "well".

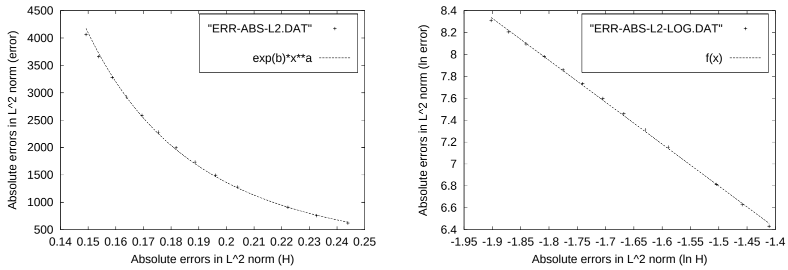

Figure 2. Errors of the solution in H^2.

We decompose domain Ω into superelements with lines which are parallel to the coordinate axes. Let distance between this lines is equal to $H = l/(2N+1)$, $N \geq 1$. So every superelement is a square with length of edge equal to H. The "well" is situated in the central superelement. Total number of superelements is equal to $K = (2N+1)^2$. We will use the Bubnov-Galerkin approximations to solve this problem.

Since we know the exact solution (7.6) and error estimate (7.5), we can obtain error estimates as a function of H. Estimates (7.5) in this case ($L = 1$, $\gamma_{i,j} = H$) give

$$\|\varphi - \varphi_h\|_X^2 \leq 4l^2 H^2 \left(\frac{1}{3}H + 1\right) \max_{\gamma_{i,j}} \alpha^2_{2,\gamma_{i,j}}. \tag{7.7}$$

Taking into account that edges $\gamma_{i,j}$ are parallel to coordinate axes, using the formula for the exact solution (7.6) and symmetry of the domain, we obtain

$$\max_{\gamma_{i,j}} \alpha_{2,\gamma_{i,j}} = \max_{\gamma_{i,j}} \max_{s \in \gamma_{i,j}} \left|\varphi^{(2)}(s)\right| \leq \max_{x \in [-l/2;+l/2]} \left|\frac{\partial^2 u(x,y)}{\partial x^2}\right|\Bigg|_{y=H/2} = \frac{4}{H^2}.$$

Finally, we have: $\|\varphi - \varphi_h\|_X^2 \leq 64\frac{l^2}{H^2}\left(\frac{1}{3}H + 1\right)$. Thus it follows that error is decreasing when distance between "well" and superelement boundary is increasing.

Analysis of numerical tests show (see Fig.2) $\|\varphi - \varphi_h\|_H^2 \sim \frac{1}{H^{1.9}}$.

The obtained strong error dependence on the distance between "well" and superelement boundary proves the importance of a priori analysis of the initial boundary value problem and its singularities.

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Fedorenko baigtinių elementų metodas kaip Galerkino metodo aproksimacija

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Pateikta vienos variacinės lygties Petrovo-Galerkino aproksimacija, kuri veda į Fedorenko baigtinių elementų metodą. Jis nagrinėjamas kaip tam tikro uždavinio Petrovo-Galerkino aproksimacija kraštinių uždavinių sprendinių pėdsakams kai kurių sričių (superelementų) kraštuose. Tokių uždavinių iteraciniai sprendimo metodai yra gerai žinomi kaip srities dekompozicijos metodai. Pateikti skaitinių eksperimentų rezultatai.