Mathematical Modelling and Analysis
Volume 22 Number 2, March 2017, 237–251
https://doi.org/10.3846/13926292.2017.1292323

Publisher: Taylor&Francis and VGTU
http://www.tandfonline.com/TMMA
ISSN: 1392-6292
eISSN: 1648-3510

Global Existence and Blow-Up for a Chemotaxis System

Xueyong Chen[a], Fuxing Hu[b], Jianhua Zhang[c] and Jianwei Shen[a]

[a] *School of Mathematics and Statistics, Xuchang University*
Xuchang, China
[b] *Department of Mathematics, Huizhou University*
Huizhou, China
[c] *School of Electrical Engineering and Automation, Jiangsu Normal University*
Xuzhou, China
E-mail(*corresp.*): xcjwshen@gmail.com

Received July 16, 2016; revised January 21, 2017; published online March 1, 2017

Abstract. In this paper we consider a Keller-Segel-type chemotaxis model with reaction term under no-flux boundary conditions, where the kinetics term of the system is power function. Assuming some growth conditions, the existence of bounded global strong solution to the parabolic-parabolic system is given. We also give the numerical test and find out that there exists a threshold. When the power frequency greater than the threshold, both global solution and blow-up solution exist.

Keywords: chemotaxis, global existence, blow-up, numerical analysis.

AMS Subject Classification: 35K55; 35Q80.

1 Introduction

Chemotaxis phenomenon is quite common phenomenon in bio-system. The first chemotaxis equation was introduced by Keller and Segel [6] to describe the aggregation of slime mold amoebae due to an attractive chemical substance.

$$\begin{cases} \dfrac{\partial u}{\partial t} = D_1 \triangle u + \nabla\cdot(\chi(w)u\nabla w), & x\in\Omega, \quad t>0,\\ \epsilon\dfrac{\partial w}{\partial t} = D_2\triangle w - kw + hu, & x\in\Omega, \quad t>0,\\ \dfrac{\partial u}{\partial n} = \dfrac{\partial w}{\partial n} = 0, & x\in\partial\Omega, \quad t>0,\\ u(x,0)=u_0(x),\ w(x,0)=w_0(x), & x\in\Omega, \end{cases}$$

where D_1 and D_2 are the diffusion coefficients of the cells u and attractant w, respectively. The function $hu - kw$ is the kinetics term of w. The classical chemotaxis model has been extensively studied in the last few years (see [2], [3], [8], [9]), Y.Tao *et al* [9] consider the global weak solution for a chemotaxis model in three-dimensional space. The ideas in the paper [9] is an effective method to deal with the chemotaxis model. In this paper the following chemotaxis model is discussed:

$$\begin{cases} \frac{\partial u}{\partial t} = \triangle u - \nabla \cdot (\frac{u}{1+\varepsilon u}\nabla w) + f(u,w), & x \in \Omega, t > 0, \\ \frac{\partial w}{\partial t} = \triangle w + g(u) - w, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} = 0, & x \in \partial\Omega, \\ u(x,0) = u_0(x), w(x,0) = w_0(x), & x \in \bar{\Omega}, \end{cases} \tag{1.1}$$

where u represents the density or population of a biological species, which could be a cell, a germ, or an insect, while w represents an attractive resource of the species. $g(u)$ is the production function of w, it is always assumed in this paper that $g(u) = u^{\gamma_0}$, $1 \le \gamma_0$. ε is positive constant, Ω is a subset of R^n. $f(u,w)$ is the reproduction function of the species, which is smooth and satisfying the conditions:

- (i) $f(0,w) = 0$;
- (ii) There exists $M > 0$ such that $f(u,w) \le 0$ as $u > M$.

It is well known that, by the argument of the local existence of the solution in [2], there exists a unique solution (u,w) of (1.1) defined on the maximal interval of its existence $[0, T_{max})$, which is smooth and nonnegative for $(x,t) \in \bar{\Omega} \times (0, T_{max})$.

Our main results in this paper are the following:

- For any bounded positive functions $0 \le u_0 \in C^0(\Omega), 0 \le w_0 \in W^{1,p}(\Omega)$, $(p > n)$, if $\gamma_0 \in [1, \frac{n+1}{n}]$, then all solutions of (1.1) are global in time and uniformly bounded. Moreover, there exists $\nu > 0$ such that

  $$\|u(t)\|_{c^\delta} + \|w(t)\|_{C^{2+\delta}} \le c(\delta)(1 + e^{-vt}(\|u_\tau\|_{L^\infty} + \|w_\tau\|_{L^2})), \quad t \ge \tau$$

  for any $\delta \in (0,1)$;

- When γ_0 slightly larger than $\frac{n+1}{n}$, the global solutions also exist in our numerical tests. So, the upper bound $\frac{n+1}{n}$ of γ_0 may be not the smallest one;

- When γ_0 is further larger than the bound $\frac{n+1}{n}$, both global solution and blow-up solution exist, which happen depend only on the initial data. It is reasonable to conjecture that there is a threshold π^*. There exists a finite time blow-up solution to system (1.1) with the initial value larger than π^*. The global solution can also exist if the smaller initial (less than π^*) conditions is proposed.

The latter two conclusions are derived by the numerical results. They are our key problems in theoretical research in the future.

2 Preliminary

For readers' convenience, some well-known inequalities and embedding results that will be used in the sequel are presented in this section.

Lemma 1. *(Gagliardo-Nirenberg inequality) If $p, q \geq 1$ and $p(n-q) < nq$, then, for $r \in (0, p)$*

$$\|u\|_{L^p(\Omega)} \leq c\|u\|^a_{W^{1,q}} \cdot \|u\|^{(1-a)}_{L^r(\Omega)}, \quad \forall u \in W^{1,q}(\Omega),$$

where $a = \dfrac{n/r - n/p}{1 - n/q + n/r} \in (0, 1)$.

Lemma 2. *(see [5]) Let $1 \leq q \leq p < \infty$ and $f \in L^q(\Omega)$. Then*

$$\|e^{t\triangle} f\|_p \leq (4\pi t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\|f\|_q, \quad \|\nabla e^{t\triangle} f\|_p \leq ct^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}}\|f\|_q,$$
$$\|(-\triangle + 1)^\beta e^{t\triangle} f\|_p \leq ct^{-\beta-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})} e^{(1-\mu)t}\|f\|_q, \quad p \neq q,$$

for all $t > 0$, where $\beta > 0, \mu > 0$ and c is a positive constant depending only on p, q, Ω. $(e^{t\triangle} f)(x) = \int_\Omega G(x-y, t) f(y) dy$ and $G(x, t)$ is the Green function of the heat equation $\frac{\partial u}{\partial t} - \triangle u = 0, x \in \Omega$ with the homogeneous Neumann boundary condition.

Lemma 3. *(see [7]) Let $A_p = -\triangle$ and $D(A_p) = \{\varphi \in W^{2,p}(\Omega) \mid \frac{\partial \varphi}{\partial n} |_{\partial\Omega} = 0\}$, if $1 < p < \infty$, then*

$$D((A_p + 1)^\beta) \hookrightarrow W^{1,p}(\Omega), \text{ if } \beta > 1/2,$$
$$D((A_p + 1)^\beta) \hookrightarrow C^\delta(\Omega), \text{ if } 2\beta - n/p > \delta \geq 0,$$
$$\|(A + 1)^\beta e^{-t(A+1)} u\|_{L^p(\Omega)} \leq ct^{-\beta}\|u\|_{L^p(\Omega)}.$$

Lemma 4. *(see [5]) Let $\beta > 0, p \in (1, +\infty)$, for all $\varepsilon > 0$, there exists $c(\varepsilon) > 0$ for any $w \in L^p(\Omega)$, such that*

$$\|(-\triangle + 1)^\beta e^{t\triangle} \nabla \cdot w\|_{L^p_{(\Omega)}} \leq c(\varepsilon) t^{-\beta-\frac{1}{2}-\varepsilon}\|w\|_{L^p_{(\Omega)}}.$$

Lemma 5. *Let r and s be nonnegative real numbers satisfying $r + s < 2$. Then for any $\varepsilon > 0$ there exists a constant $c_\varepsilon > 0$ such that*

$$a^r b^s \leq \varepsilon(a^2 + b^2) + c_\varepsilon, \qquad \forall a, b > 0.$$

3 Global existence and some priori estimates

In this section, the global-in-time existence of a solution to system (1.1) is proved. The following priori estimates will play a crucial role in the proof of our result.

Lemma 6. *Suppose that* $0 \leq u_0 \in L^2(\Omega)$, (u,w) *is the solution of (1.1), then there exists a positive constant* c_0 *such that*

$$\|u(t)\|_1 \leq c_0, \qquad \forall t > 0.$$

Proof. Integrating the fist equation of (1.1) on Ω, and by the boundary conditions, we obtain

$$\frac{d\|u(t)\|_1}{dt} = \int_\Omega f(u,w)dx.$$

From the condition (ii) in the Section 1, it is easy to know that $\frac{d\|u(t)\|_1}{dt} < 0$ as $\|u(t)\|_1 \geq M|\Omega|$. Then there exists a constant c_0 (depending only on M, Ω) such that $\|u(t)\|_1 \leq c_0$ for all $t > 0$. $\square$

Lemma 7. *Suppose that* $1 \leq \gamma_0 \leq \frac{n+1}{n}$, *for any non-negative* $u_0 \in C^0(\Omega), w_0 \in W^{1,p}(\Omega)(p > n)$ *there exist* $T \in [0,+\infty)$ *and one pair non-negative function satisfying*

$$u \in C^0(\Omega \times [0,T)) \bigcap C^{2,1}((\Omega) \times (0,T)),$$
$$w \in C^0(W^{1,p}(\Omega);[0,T)) \bigcap C^{2,1}((\Omega) \times (0,T))$$

such that (u,w) *solves (1.1) in the classical sense. Moreover,*

$$\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 \leq c(1 + e^{-vt}(\|u_0\|_{L^2}^2 + \|w_0\|_{L^2}^2)), \quad 0 \leq t < T,$$

where $v > 0$.

Proof. The local existence of the system (1.1) has been proved by N.Bellomo *et al* in [1] and the local solution (u,w) satisfy

$$u \in C^0(\Omega \times [0,T)) \bigcap C^{2,1}((\Omega) \times (0,T)),$$
$$w \in C^0(W^{1,p}(\Omega);[0,T)) \bigcap C^{2,1}((\Omega) \times (0,T)).$$

Now, the uniformly bounded estimate will be given. Multiplying the first equation of (1.1) by u and integrate the product in Ω. Then

$$\begin{aligned}\frac{1}{2}\frac{d\|u(t)\|_{L^2}^2}{dt} + \int_\Omega |\nabla u|^2 dx &= \int_\Omega \frac{u}{1+\varepsilon u}\nabla u \cdot \nabla w dx + \int_\Omega u(au - bu^2)dx \\ &\leq \frac{1}{\varepsilon}\int_\Omega |\nabla u \cdot \nabla w| dx + \frac{a^2}{4b}\int_\Omega u dx. \qquad (3.1)\end{aligned}$$

Taking the inner product of the second equation to (1.1) with w in $L^2(\Omega)$, we find that

$$\begin{aligned}\frac{1}{2}\frac{d\|w(t)\|_{L^2}^2}{dt} + \int_\Omega |\nabla w|^2 dx &= -\int_\Omega |w|^2 dx + \int_\Omega u^{\gamma_0} w dx \\ &\leq -\frac{1}{2}\int_\Omega |w|^2 dx + \frac{1}{2}\int_\Omega u^{2\gamma_0} dx. \qquad (3.2)\end{aligned}$$

From Lemma 1 and $1 \le \gamma_0 \le \frac{n+1}{n}$, there exists $a = \frac{2n(1-1/(2\gamma_0))}{n+2} \in (0,1)$, such that

$$\|u\|_{L^{2\gamma_0}} \le \|u\|^a_{W^{1,2}} \|u\|^{1-a}_{L^1}, \quad 2\gamma_0 a = \frac{2n(2\gamma_0 - 1)}{n+2} < \frac{2n(2\frac{n+1}{n} - 1)}{n+2} = 2.$$

By Hölder inequality, Poincaré inequality and Lemma 6 then

$$\|u\|^{2\gamma_0}_{L^{2\gamma_0}} \le c\|u\|^{2\gamma_0 a}_{W^{1,2}} \le \zeta \|\nabla u\|^2_{L^2} + C_\zeta,$$

where $\zeta > 0$ is an arbitrary constant and C_ζ is a constant depend on ζ. From above analysis and (3.1), (3.2), there is $\upsilon = \min\{1, 2 - \zeta - 1/\varepsilon\} > 0$ such that

$$\frac{d}{dt}(\|w(t)\|^2_{L^2} + \|u(t)\|^2_{L^2}) \le -\upsilon(\|w(t)\|^2_{L^2} + \|u(t)\|^2_{L^2}) + C_\zeta.$$

By the Gronwall's Lemma

$$\|u(t)\|^2_{L^2} + \|w(t)\|^2_{L^2} \le c(1 + e^{-\upsilon t}(\|u_0\|^2_{L^2} + \|w_0\|^2_{L^2})), \quad t > 0.$$

□

Lemma 8. *Assume that* $1 \le q \le p \le \infty$, $\frac{1}{q} - \frac{1}{p} < \frac{1}{n}$ *and* $u \in L^\infty([0,\infty); L^{q\gamma_0}_{(\Omega)})$, *then for any* $\tau > 0$

$$\|w(t)\|_{W^{1,p}} \le \tau^{-\alpha}\|w_0\|_{L^1} + c\Gamma(\gamma) \sup_{\tau < s < t} \|u(s)\|^{\gamma_0}_{q\gamma_0}, \quad \forall t \ge \tau,$$

where $\alpha > 0$, $\gamma = 1 - \beta - \frac{n}{2}(\frac{1}{q} - \frac{1}{p})$ *and*

$$w(x,t) = e^{t(\triangle - 1)}w_0 + \int_0^t e^{(t-s)(\triangle-1)}u^{\gamma_0}(s)ds.$$

Proof. Since $\frac{1}{q} - \frac{1}{p} < \frac{1}{n}$, there exist some $\beta > \frac{1}{2}$ such that

$$\frac{1}{q} - \frac{1}{n} + \frac{2}{n}(\beta - \frac{1}{2}) < \frac{1}{p}.$$

Multiplying the second equation in (1.1) by e^t, then

$$\frac{\partial}{\partial t}(e^t w) = \triangle(e^t w) + e^t u^{\gamma_0}.$$

Applying $(A+1)^\beta$ to both sides of the representation formula of $e^t w$, by Lemma 2–3, we have

$$\begin{aligned}
\|(A+1)^\beta e^t w(t)\|_{L^p} &\le \int_0^t (A+1)^\beta e^{-(t-s)A}\|e^s u^{\gamma_0}(s)\|_p ds \\
&\quad + t^{-\beta - \frac{n}{2}(1-\frac{1}{p})} e^{(1-\mu)t}\|(A+1)^\beta e^{-tA}w(0)\|_{L^p} \\
&\le c\int_0^t (t-s)^{-\beta-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})} e^{(1-\mu)(t-s)}\|e^s u^{\gamma_0}(s)\|_q ds \\
&\quad + ct^{-\beta-\frac{n}{2}(1-\frac{1}{p})} e^{(1-\mu)t}\|w(0)\|_{L^1},
\end{aligned}$$

then

$$\|w(t)\|_{W^{1,p}} \leq \|(A+1)^{\beta} w(t)\|_{L^p} \leq c \int_0^t (t-s)^{-\beta-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})} e^{-\mu(t-s)} \|u^{\gamma_0}(s)\|_q ds$$
$$+ ct^{-\beta-\frac{n}{2}(1-\frac{1}{p})} \|w(0)\|_{L^1} \leq c \sup_{0<s<t} \|u^{\gamma_0}(s)\|_q \int_0^{\infty} e^{-s} s^{\gamma-1} ds$$
$$+ c\tau^{-\beta-\frac{n}{2}(1-\frac{1}{p})} \|w(0)\|_{L^1} = c\Gamma(\gamma) \sup_{0<s<t} \|u(s)\|_{q\gamma_0}^{\gamma_0} + c\tau^{-\alpha} \|w(0)\|_{L^1},$$

here $\alpha = \beta + \frac{n}{2}(1-\frac{1}{p})$, $\gamma = 1 - \beta - \frac{n}{2}(\frac{1}{q}-\frac{1}{p})$. □

Lemma 9. *Assume that u and w satisfy the estimates*

$$\|u(t)\|_{L^{q_0}} \leq c, \quad \|\nabla w(t)\|_{L^{\gamma}} \leq c, \quad \forall t \in [\tau, T],$$

here $(\frac{n}{\gamma}-1)q_0 < n$, $q_0 \geq 1$ and $\gamma > 2$. Then for any $q > \max\{q_0, q_0 + 2 - \frac{2q_0}{\gamma}\}$ which fulfils $(\frac{n}{\gamma}-1)q < n-2$, there exists a constant $c(q)$ such that

$$\|u(t)\|_{L^q} \leq c(q), \quad \forall t \in [\tau, T).$$

Proof. Taking the inner product of the first equation to (1.1) with u^{q-1} and writing $v = u^{\frac{q}{2}}$, then for any $t \in [\tau, T)$

$$\frac{d}{dt}\int_{\Omega} |v|^2 dx + \frac{2(q-1)}{q} \int_{\Omega} |\nabla v|^2 dx \leq \frac{q-1}{2\varepsilon^2} \int_{\Omega} v^{2-\frac{4}{q}} |\nabla w|^2 dx + \delta_0 \int_{\Omega} v^2 dx + c_{\delta_0}$$
$$\leq \frac{q-1}{2\varepsilon^2} \|v\|_{L^{\frac{2p_1(q-2)}{q}}}^{\frac{2(q-2)}{q}} \|\nabla w\|_{L^{\gamma}}^2 + \delta_0 \int_{\Omega} v^2 dx + c_{\delta_0},$$

where $p_1 = \frac{\gamma}{\gamma-2}$. Since $\|u(t)\|_{L^{q_0}} \leq c$, $(\frac{n}{\gamma}-1)q < n-2$ and $q > q_0 + 2 - \frac{2q_0}{\gamma}$, by the Gagliardo-Nirenberg and the Poincaré inequality

$$\|v\|_{L^{\frac{2p_1(q-2)}{q}}}^{\frac{2(q-2)}{q}} \leq c\|v\|_{W^{1,2}}^{a\frac{2(q-2)}{q}} \|v\|_{L^{\frac{2q_0}{q}}}^{(1-a)\frac{2(q-2)}{q}} \leq c\|\nabla v\|_{L^2}^{a\frac{2(q-2)}{q}},$$

where $a = \frac{(nq/2q_0 - nq(\gamma-2)/(2\gamma(q-2))}{1-n/2+nq/(2q_0)} \in (0,1)$. Since $(\frac{n}{\gamma}-1)q_0 < n$ means that

$$a\frac{2(q-2)}{q} = \frac{\frac{n(q-2)}{q_0} - \frac{n(\gamma-2)}{\gamma(q-2)}}{1-n/2+nq/(2q_0)} < 2,$$

by Young's inequality, for any $\varepsilon_0 > 0$, there exists a constant c_{ε_0} such that

$$\frac{q-1}{2\varepsilon^2} \|v\|_{L^{\frac{2p_1(q-2)}{q}}}^{\frac{2(q-2)}{q}} \leq \varepsilon_0 \|\nabla v\|_{L^2}^2 + c_{\varepsilon_0}.$$

Let ε_0, δ_0 small enough and by the Poincaré inequality, there exists $\nu(q) > 0$ such that

$$\frac{d}{dt}\int_{\Omega} |v|^2 dx + \frac{2(q-1)}{q} \int_{\Omega} |\nabla v|^2 dx \leq -(\frac{2(q-1)}{q} - \varepsilon_0) \int_{\Omega} |\nabla v|^2 dx$$
$$+ \delta_0 \int_{\Omega} v^2 dx + c_{\delta_0} \leq -\nu(q) \int_{\Omega} v^2 dx + c_{\delta_0} + c_{\varepsilon_0}, \quad \forall t \in [\tau, T).$$

In view of Gronwall's Lemma, this shows that

$$\int_\Omega |u(t)|^q dx \le c(c_{\varepsilon_0,\delta_0} + \|u(\tau)\|_{L^q} e^{-\nu(q)(t-\tau)}).$$

□

Lemma 10. *Suppose that* $0 \le u_0 \in C^0(\Omega), 0 \le w_0 \in W^{1,p}(\Omega)(p > n)$, (u, w) *is a local solution of (1.1) in* $[0, T)$ *satisfying*

$$u \in C^0([0,T); C^0(\Omega)) \bigcap C^{2,1}((\Omega); (0,T)),$$
$$w \in C^0([0,T); W^{1,p}(\Omega)) \bigcap C^{2,1}((\Omega); (0,T)) \bigcap C^0((0,T); C^3(\Omega)),$$

then for any $q > 2\gamma_0$

$$\|u(t)\|^2_{L^q} + \|w(t)\|^2_{W^{1,2}} \le c(1 + e^{-\upsilon t}(\|u_0\|^2_{L^q} + \|w_0\|^2_{W^{1,2}})), 0 \le t < T.$$

Proof. Taking the inner product of the first equation to (1.1) with u^{q-1} in $L^2(\Omega)$, we find that

$$\begin{aligned}
&\frac{1}{q}\frac{d}{dt}\int_\Omega |u|^q dx + (q-1)\int_\Omega u^{q-2}|\nabla u|^2 dx \\
&\quad = (q-1)\int_\Omega \frac{u^{q-1}}{1+\varepsilon u}\nabla u\,\nabla w dx + \int_\Omega u^{q-1} f(u,w) dx \\
&\quad \le \frac{(q-1)\varepsilon^2}{2}\int_\Omega \frac{u^q}{(1+\varepsilon u)^2}|\nabla u|^2 dx + \frac{a^2}{4b}\int_\Omega u^{q-1} dx + \frac{q-1}{2\varepsilon^2}\int_\Omega u^{q-2}|\nabla w|^2 dx.
\end{aligned}$$

Let $v = u^{\frac{q}{2}}$, then

$$\begin{aligned}
\frac{d}{dt}\int_\Omega |v|^2 dx + \frac{4(q-1)}{q}\int_\Omega |\nabla v|^2 dx &\le \frac{2(q-1)}{q}\int_\Omega |\nabla v|^2 dx + \delta_0 \int_\Omega v^2 dx \\
&\quad + \frac{q-1}{2\varepsilon^2}\int_\Omega v^{2-\frac{4}{q}}|\nabla w|^2 dx + c_{\delta_0},
\end{aligned} \tag{3.3}$$

where $\delta_0 > 0$ is an arbitrary constant and c_{δ_0} depends on a, b, Ω, δ_0. By Hölder's inequality, for any $p_1 > 1$ and $\frac{1}{p_1} + \frac{1}{p_1'} = 1$

$$\int_\Omega v^{2-\frac{4}{q}}|\nabla w|^2 dx \le \|v\|^{\frac{2(q-2)}{q}}_{L^{\frac{2p_1(q-2)}{q}}} \|\nabla w\|^2_{L^{2p_1'}}.$$

Let $\alpha = \frac{2(q-2)}{q}$, using the Gagliardo-Nirenberg inequality and Poincaré inequality, if $(q-2)p_1 > 2$ and $\alpha p_1(n-2) < 2n$, we can obtain that

$$\|v\|^\alpha_{L^{\alpha p_1}} \le c\|v\|^{a\alpha}_{W^{1,2}}\|v\|^{(1-a)\alpha}_{L^{\frac{4}{q}}} \le c\|\nabla v\|^{a\alpha}_{L^2},$$

where $a = \frac{nq/4 - n/(\alpha p_1)}{1-n/2+nq/4} \in (0,1)$, $\|v\|_{L^{\frac{4}{q}}} = \|u\|_{L^2} \le c$ for $0 \le t \le T$.

From Lemma 8 (applied to $q\gamma_0 = 2$), for any $\gamma \in [\frac{2}{\gamma_0}, \frac{1}{\gamma_0/2-1/n})$ there exists a constant c such that $\|\nabla w\|_{L^\gamma} < c, \forall t \in [\tau, T)$. Since $1 \le \gamma_0 \le 1 + 1/n$, hence we can choose $\gamma = 2$. Employing Poincaré inequality and Lemma 1 again,

$$\|\nabla w\|^2_{L^{2p_1'}} \le c\|\nabla w\|^{2a_1}_{W^{1,2}}\|\nabla w\|^{2(1-a_1)}_{L^2} \le c\|\triangle w\|^{2a_1}_{L^2},$$

where $p_1'(n-2) < n$ and $a_1 = \frac{n}{2} - \frac{n}{2p_1'} \in (0,1)$. As a result, we see that

$$\int_\Omega v^{2-\frac{4}{q}}|\nabla w|^2 dx \le c\|\nabla v\|^{a\alpha}_{L^2} \cdot \|\triangle w\|^{2a_1}_{L^2}$$

for all $t \in [\tau, T)$. Suppose that $\frac{1}{p_1} = s$, then $2a_1 + a\alpha = \frac{n^2(p-2)s+2n(p-2)}{4+n(p-2)}$.

Step 1. Let us first consider the case $n = 1, 2$. In this case, for any $q > 2\gamma_0$ there exists a constant $s < \min\{1, \frac{8}{n^2(p-2)}\}$ such that

$$2a_1 + a\alpha = \frac{n^2(p-2)s + 2n(p-2)}{4 + n(p-2)} < 2.$$

From Lemma 5, for any $\delta > 0$ there exists a constant $c_\delta > 0$ such that

$$\int_\Omega v^{2-\frac{4}{q}}|\nabla w|^2 dx \le \delta(\|\nabla v\|^2_{L^2} + \|\triangle w\|^2_{L^2}) + c_\delta.$$

Taking the inner product of the second equation to (1.1) with $\triangle w$ in $L^2(\Omega)$, we find that

$$\begin{aligned}\frac{1}{2}\frac{d\|\nabla w(t)\|^2_{L^2}}{dt} &+ \int_\Omega |\triangle w|^2 dx + \int_\Omega |\nabla w|^2 dx = -\int_\Omega u^{\gamma_0}\triangle w dx \\ &\le \frac{1}{2}\int_\Omega u^{2\gamma_0}dx + \frac{1}{2}\int_\Omega |\triangle w|^2 dx \le \delta_1 \int_\Omega u^q dx + c_{\delta_1} + \frac{1}{2}\int_\Omega |\triangle w|^2 dx, \quad (3.4)\end{aligned}$$

where $\delta_1 > 0$ is small enough and $c_{\delta_1} > 0$ is a constant depending on $q, \delta_1, \Omega, \gamma_0$.

The standard Poincaré inequality ensures that there exist positive constants c_1 and c_2 such that

$$\int_\Omega |\nabla v|^2 dx \ge c_1 \int_\Omega |v|^2 dx \quad \text{and} \quad \int_\Omega |\triangle w|^2 dx \ge c_2 \int_\Omega |\nabla w|^2 dx.$$

From (3.3), (3.4) and above estimates, there exist small enough $\delta, \delta_0, \delta_1$ such that $\nu = \min\{c_1(\frac{2q-2}{q} - \frac{(q-1)\delta}{2\varepsilon^2}) - \delta_0 - 2\delta_1, \quad c_2(1 - \frac{(q-1)\delta}{2\varepsilon^2}) + 2\} > 0$,

$$\frac{d(\|\nabla w(t)\|^2_{L^2} + \|v(t)\|^2_{L^2})}{dt} + \nu\left(\int_\Omega |\nabla w|^2 dx + \int_\Omega |v|^2 dx\right) \le C,$$

where C is a constant depending on $c_\delta, c_{\delta_0}, c_{\delta_1}, \Omega$. That Gronwall's Lemma yields

$$\|v(t)\|^2_{L^2} + \|w(t)\|^2_{W^{1,2}} \le c(1 + e^{-\nu t}(\|v_\tau\|^2_{L^2} + \|w_\tau\|^2_{W^{1,2}})) \ , t \in [\tau, T),$$

which implies that for any $q > 2\gamma_0$, $\|u(t)\|_{L^q}$ is bounded. By Lemma 8, for any $\gamma > 0$, $\|w(t)\|_{W^{1,\gamma}}$ is bounded too.

Step 2. In the case $n \geq 3$, for any $2\gamma_0 \leq q < \frac{2n}{n-2}$ there exists $s \in (\frac{(n-2)(q-2)}{nq}, \frac{8}{n^2(q-2)})$ such that $2a_1 + a\alpha = \frac{n^2(p-2)s+2n(p-2)}{4+n(p-2)} < 2$. With the same analysis as in step 1, for any $q \in [2\gamma_0, \frac{2n}{n-2})$ there exists $\nu_1 > 0$ such that

$$\int_\Omega |u|^q dx \leq c(1 + e^{-\nu_1 t}(\|u_\tau\|^q_{L^q} + \|w_\tau\|^2_{W^{1,2}})) \quad t \in [\tau, T). \tag{3.5}$$

In order to prove above estimates is right for arbitrarily large q, we have to apply Lemmas 8–9 several times. First, we define a sequence numbers a_k by

$$a_0 := q_0 \quad \text{and} \quad a_k = \begin{cases} \frac{n-2}{n-a_{k-1}} a_{k-1}, & \text{if} \quad a_{k-1} < n, \\ +\infty, & \text{else}, \end{cases}$$

where $q_0 > \max\{2, \frac{2n}{n+2}\gamma_0\}$ and $a_k > 2$ (which means that a_k is a monotone increasing sequence for all k satisfying $a_k < n$). Now, we prove that there exists $k_0 \in N$ such that $a_{k_0} = +\infty$. Otherwise, a_k is a drab bounded sequence, hence there exists the limit $a_k \to a_\infty \leq n$ as $k \to \infty$ and $a_\infty = \frac{n-2}{n-a_\infty} a_\infty$, this means $a_\infty = 2 < q_0$ contradicting the monotonicity. Therefore $a_{k_0} = +\infty$ for some $k_0 \in N$.

By a continuity argument, it is possible to choose positive $\epsilon_1, ...\epsilon_{k_0}$ such that $q_1, ...q_{k_0}$ defined by

$$q_k = \frac{n-2}{n-q_{k-1}} q_{k-1} - \epsilon_k, \quad k = 1, ..., k_0,$$

satisfying $q_0 < q_1 < ... < q_{k_0-1} < n$ and $q_{k_0} > q$. Now for $k = 1, ...k_0$,

$$\overline{p}_{k-1} = \frac{nq_{k-1}}{n\gamma_0 - q_{k-1}}.$$

Since $q_{k-1} \geq q_0 > \frac{2n}{n+2}\gamma_0$,

$$\overline{p}_{k-1} - 2 = \frac{(n+2)q_{k-1} - 2n\gamma_0}{n\gamma_0 - q_{k-1}} > 0.$$

Furthermore, since $\gamma_0 \in [1, \frac{n+1}{n})$

$$\left(\frac{n}{\overline{p}_{k-1}} - 1\right) q_{k-1} < n\gamma_0 - 2q_{k-1} < \gamma_0 \frac{n(n-2)}{n+2} \leq n.$$

Finally, because $q_{k-1} > 1 > n(\gamma_0 - 1)$

$$\left(\frac{n}{\overline{p}_{k-1}} - 1\right) q_k < \left(\frac{n}{\overline{p}_{k-1}} - 1\right) q_{k-1} \frac{n-2}{n-q_{k-1}} < (n-2)\frac{n\gamma_0 - 2q_{k-1}}{n - q_{k-1}} < n-2.$$

From above analysis, it is possible to fix $p_0, ..., p_{k_0-1}$ such that $2 < p_{k-1} < \overline{p}_{k-1}$,

$$(n/p_{k-1} - 1)\, q_{k-1} < n \quad \text{and} \quad (n/p_k - 1)\, q_{k-1} < n-2$$

for all $k = 1, ..., k_0$. Furthermore, we choose any sequence of numbers $\frac{\tau}{2} < \tau_0 < \tau_1 < ... < \tau_{k_0} = \tau$. We now claim that for any $k = 0, ..., k_0$,

$$\int_\Omega |u|^{q_k} dx \leq c(1 + e^{-\nu_k t}(\|u_\tau\|^q_{L^q} + \|w_\tau\|^2_{W^{1,2}})), t \in [\tau_k, T) \tag{3.6}$$

for appropriate $\nu_k > 0$. In the case $k = 0$, (9) is implied by (3.5). However, if

$$\int_\Omega |u|^{q_{k-1}} dx \leq c(1 + e^{-\nu_{k-1} t}(\|u_\tau\|^q_{L^q} + \|w_\tau\|^2_{W^{1,2}})), t \in [\tau_{k-1}, T)$$

holds for some $k \in \{1, ..., k_0\}$, since $2 < p_{k-1} < \overline{p}_{k-1} = \frac{nq_{k-1}}{n\gamma_0 - q_{k-1}}$ Lemma 8 yields

$$\|\nabla w(t)\|_{L^{p_{k-1}}} \leq c, t \in [\tau_k, T).$$

From $(\frac{n}{p_{k-1}} - 1)q_{k-1} < n$ and $(\frac{n}{p_k} - 1)q_{k-1} < n - 2$ (3.6) is proved by Lemma 9. Then for any arbitrarily large q there exists $\nu > 0$ such that

$$\int_\Omega |u|^q dx \leq c(q, \tau)(1 + e^{-\nu t}(\|u_\tau\|^q_{L^q} + \|w_\tau\|^2_{W^{1,2}})), t \in [\tau, T).$$

□

In above analysis we denote any positive constant by c which may change from line to line. By Lemma 10 and all of the estimates above, the global existence of the solution to (1.1) is obtained as the following theorem.

Theorem 1. *For any bounded positive functions* $0 \leq u_0 \in C^0(\Omega), 0 \leq w_0 \in W^{1,p}(\Omega)(p > n)$, *if* $\gamma_0 \in [1, \frac{n+1}{n}]$, *then all solutions of (1.1) are global in time and uniformly bounded. Moreover, the solution satisfies*

$$0 \leq u \in C([0, \infty); L^2(\Omega)) \bigcap C^1((0, \infty); L^2(\Omega)) \bigcap C((0, \infty); C^\delta(\Omega)),$$
$$0 \leq w \in C([0, \infty); W^{1,p}(\Omega)) \bigcap C^1((0, \infty); W^{1,p}(\Omega)) \bigcap C((0, \infty); C^{2+\delta}(\Omega))$$

and there exists $\nu > 0$ *such that*

$$\|u(t)\|_{c^\delta} + \|w(t)\|_{C^{2+\delta}} \leq c(\delta)(1 + e^{-vt}(\|u_\tau\|_{L^\infty} + \|w_\tau\|_{L^2})), t \geq \tau$$

for some $\delta \in (0, 1)$.

Proof. From Lemma 3, for any $\beta \in (0, \frac{1}{2})$ and $p > 1$ satisfying $2\beta - n/p > \delta$,

$$D((A_p + 1)^\beta) \hookrightarrow C^\delta(\Omega) \quad \text{and} \quad D((A_p + 1)^{1+\beta}) \hookrightarrow C^{2+\delta}(\Omega). \tag{3.7}$$

In order to prove the results, we fix β and apply $(A_p + 1)^\beta$ to both sides of the formula

$$u(t) = e^{-(t-\frac{\tau}{2})(A+1)} u_{\frac{\tau}{2}} - \int_{\frac{\tau}{2}}^t e^{-(t-s)(A+1)} [\nabla(\frac{u}{1+\varepsilon u} \nabla w) + f(u) + u] ds, t \in (\frac{\tau}{2}, T).$$

By Lemma 8 and Lemma 10, then

$$\|u(t)\|_{L^{p\gamma_0}} + \|w(t)\|_{W^{1,p}} \leq c(\delta)(1 + e^{-vt}(\|u_\tau\|_{L^\infty} + \|w_\tau\|_{L^2})), t \in [\tau/2, T).$$

Using this, Lemma 4 and the Hölder inequality,

$$\begin{aligned}\|(A_p+1)^{\beta}u(t)\|_{L^p} &\le \|(A_p+1)^{\beta}e^{-(t-\frac{\tau}{2})(A+1)}u_{\frac{\tau}{2}}\|_{L^p}\\ &+\int_{\frac{\tau}{2}}^{t}\|(A+1)^{\beta}e^{-(t-s)(A+1)}\nabla(\frac{u}{1+\varepsilon u}\nabla w)\|_{L^p}ds\\ &+\int_{\frac{\tau}{2}}^{t}\|(A+1)^{\beta}e^{-(t-s)(A+1)}[f(u(s)+u])\|_{L^p}ds\\ &\le c(\tau,\beta,p)+c\int_{\frac{\tau}{2}}^{t}(t-s)^{-\frac{1}{2}-\beta-\epsilon_0}e^{-\mu(t-s)}\|\nabla w\|_{L^p}ds\\ &\le c(\tau,\beta,p)(1+e^{-\upsilon t}(\|u_\tau\|_{L^\infty}+\|w_\tau\|_{L^2}))\quad t\in[3\tau/4,T),\end{aligned}\tag{3.8}$$

where $\epsilon_0\in(0,\frac{1}{2}-\beta)$ and $\nu>0,\upsilon_1>0$ small enough.

Now applying $(A_p+1)^{\beta+1}$ to both sides of

$$w(t)=e^{-(t-\frac{3\tau}{4})(A+1)}w_{\frac{3\tau}{4}}+\int_{\frac{3\tau}{4}}^{t}e^{-(t-s)(A+1)}u^{\gamma_0}(s)ds,t\in[\frac{3\tau}{4},T).$$

Then, for any fixed $\epsilon_0\in(0,\frac{1}{2}-\beta)$,

$$\begin{aligned}\|(A_p+1)^{\beta+1}w(t)\|_{L^p} &\le \|(A_p+1)^{\beta+1}e^{-(t-\frac{3\tau}{4})(A+1)}w_{\frac{3\tau}{4}}\|_{L^p}\\ &+\int_{\frac{3\tau}{4}}^{t}\|(A_p+1)^{\beta+1}e^{-(t-s)(A+1)}u^{\gamma_0}(s)\|_{L^p}ds\\ &\le c(t-\frac{3\tau}{4})^{-1-\beta-\frac{n}{2}(1-\frac{1}{p})}\|w(\frac{3\tau}{4})\|_{L^1}\\ &+c\int_{\frac{3\tau}{4}}^{t}(t-s)^{1-\epsilon_0}e^{-\mu(t-s)}\|(A+1)^{\beta+\epsilon_0}u^{\gamma_0}(s)\|_{L^p}ds\\ &\le c(\tau,\beta,p)(1+e^{-\upsilon t}(\|u_\tau\|_{L^\infty}+\|w_\tau\|_{L^2})),t\in[\tau,T).\end{aligned}\tag{3.9}$$

From (3.7), (3.8) and (3.9), for any $\delta<2\beta-\frac{n}{p}\in(0,1)$, there exists $\nu>0$ such that

$$\|u(t)\|_{c^\delta}+\|w(t)\|_{C^{2+\delta}}\le c(\delta)(1+e^{-\upsilon t}(\|u_\tau\|_{L^\infty}+\|w_\tau\|_{L^2})),t\in[\tau,T).$$

Furthermore,

$$\|u(t)\|_{L^\infty}+\|w(t)\|_{L^\infty}\le c(1+e^{-\upsilon t}(\|u_\tau\|_{L^\infty}+\|w_\tau\|_{L^2})),t\in[\tau,T).$$

From above estimate, if $T<\infty$, then the solution $(u(x,t),w(x,t))$ is the bounded smooth function in $\Omega\times[0,T)$, which can be extended continuously on $\Omega\times[0,T]$. Thus we replace the initial functions $(u_0(x),w_0(x))$ by $(u(x,T),w(x,T))$, which means, from the existence of the local in time solution of (1.1), that there exists $d>0$ such that the solution $(u(x,t),w(x,t))$ of (1.1) can be extended continuously to $[0,T+d)$. The same process can be repeated, which implies that the solution $(u(x,t),w(x,t))$ of (1.1) can be extended continuously to $[0,\infty)$. Which implies that the solution to (1.1) is global-in-time and

$$\|u(t)\|_{c^\delta}+\|w(t)\|_{C^{2+\delta}}\le c(\delta)(1+e^{-\upsilon t}(\|u_\tau\|_{L^\infty}+\|w_\tau\|_{L^2})),t>\tau.$$

□

Remark 1. In this section, the existence and uniformly bounded of the solution to system (1.1) on the condition that $\gamma_0 \in [1, \frac{n+1}{n}]$ are obtained. From the proses of the proof, it is easy to know that the similar results can be obtained for $u \le g(u) \le u^{\gamma_0}$ as $u \ge 1$. In fact, we wonder if the upper bound $\frac{n+1}{n}$ of γ_0 is the best one, and what will happen when $\gamma_0 > \frac{n+1}{n}$. So we give the numerical analysis to the system in next section.

4 Numerical tests and Conclusions

For $\gamma_0 = 1$, $n = 3$, Winkler [10], [11] prove that for any prescribed total mass of cells, there exist radially symmetric positive initial data such that the corresponding solution blows up in finite time. Dynamics and sensitivity of the solutions for a bacterial self-organization model is investigated in [4]. Since the existence of cell kinetics term $f(u, w)$ and without the assumption of radially symmetric in this paper, we can't construct effective energy function as done in [5], [11]. Our future research is to study the blow-up issue of the system (1.1). In this section several 1D and 2D numerical simulations are used to validate Theorem 1, i.e. the solutions to (1.1) are global in time when $\gamma_0 \in [1, 2]$ and $[1, 1.5]$ for 1D and 2D problems, respectively. The numerical results show that blow-up solution exist for $\gamma_0 > \frac{n+1}{n}$, here$f(u, w) = u(1 - u)w$.

4.1 1D simulations

Our first example considers the problem with large initial states

$$u(x, 0) = 2, \quad w(x, 0) = 100x(1 - x),$$

where $x \in [0, 1]$.

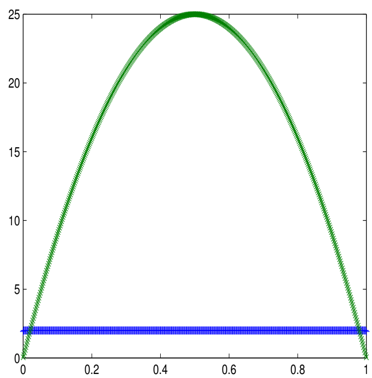

Figure 1. $\gamma_0 = 1.9$, the structure of solutions at $t = 0$.

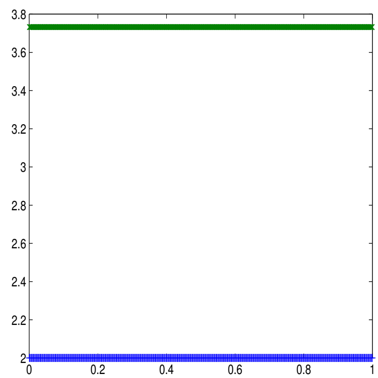

Figure 2. $\gamma_0 = 1.9$, the structure of solutions at $t = 10$.

In Figure 1 and Figure 2, the parameter γ_0 is chosen to be 1.9, which lies in the required interval $[1, 2]$ in Theorem 1, and we show the initial states at

$t = 0$ and solution at $t = 10$. For this initial value problem, the solutions arrive the steady states around $t = 10$ in which the solution varies not with time and spatial coordinates. These above conclusions are coincide with the Theorem 1.

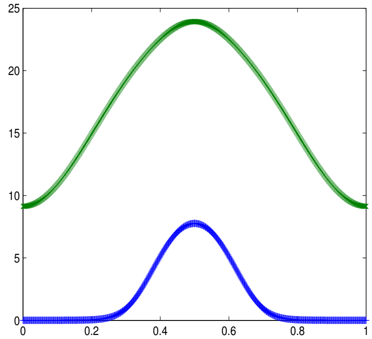

Figure 3. $\gamma_0 = 3$, the structure of solutions at $t = 0.1$.

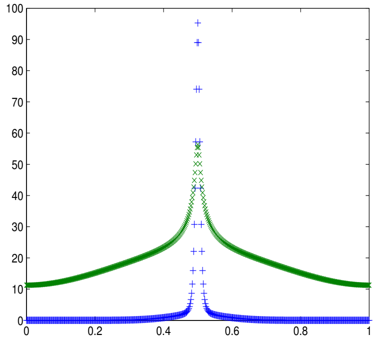

Figure 4. $\gamma_0 = 3$, the structure of solutions at $t = 0.178$.

Next, Figure 3 and Figure 4 show the solutions with $\gamma_0 = 3$ which doesn't lie in $[1, 2]$. A blow-up solution is observed around $t = 0.18$.

4.2 2D simulations

Our 2D simulations consider the following initial conditions

$$u(x, y, 0) = 2, \quad w(x, y, 0) = 100(0.5 - (x - 0.5)^2 - (y - 0.5)^2), \tag{4.1}$$

where $(x, y) \in [0, 1] \times [0, 1]$. Also, Figure 3 shows the steady-state solutions around $t = 5$ with $\gamma_0 = 1.4$ which lies in the required interval $[1, 1.5]$.

In Figure 5 and Figure 6, where only the density $u(x, t)$ is shown, when $\gamma_0 = 2$ lies out of $[1, 1.5]$, a similar blow-up solution appears around $t = 0.05$.

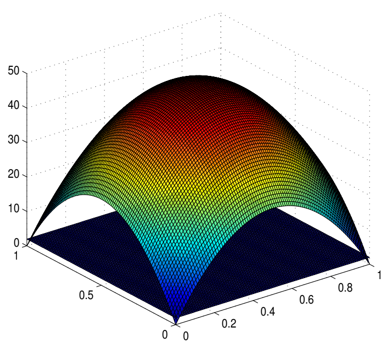

Figure 5. $\gamma_0 = 1.4$, the structure of solutions at $t = 0$.

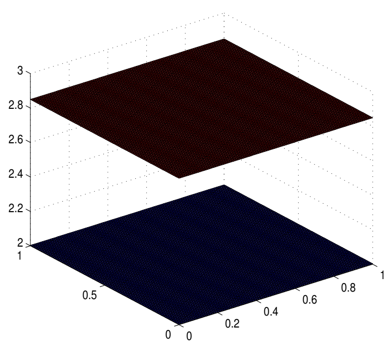

Figure 6. $\gamma_0 = 1.4$, the structure of solutions at $t = 5$.

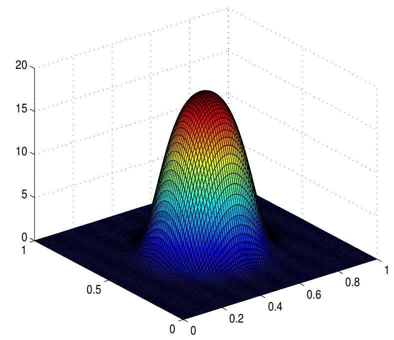

Figure 7. $\gamma_0 = 2$, the structure of density $u(x,t)$ is shown at $t = 0.01$.

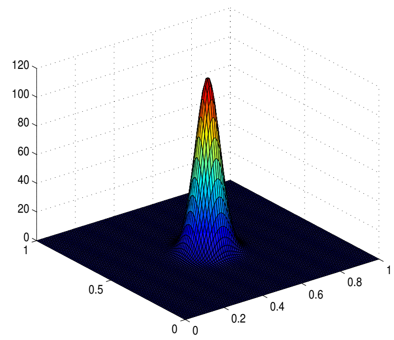

Figure 8. $\gamma_0 = 2$, the structure of density $u(x,t)$ at $t = 0.04$.

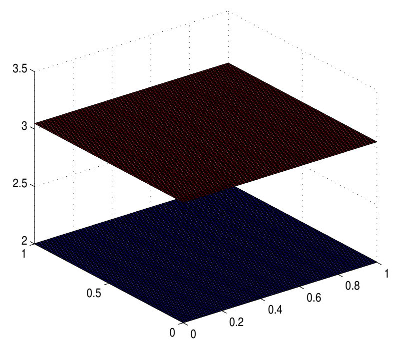

Figure 9. $\gamma_0 = 1.6$ under the initial conditions (4.1) at $t = 8$.

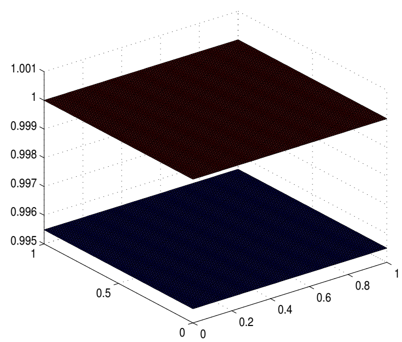

Figure 10. $\gamma_0 = 2$ under the initial conditions $u(x,y,0) = 1, u(x,y,0) = (0.5-(x-0.5)^2-(y-0.5)^2)$ at $t = 5$.

Through the above several tests, we have verified the Theorem 4.1 numerically. In addition, the following points should be clarified. First, we definitely have a global solution when $\gamma_0 \in [1, \frac{n+1}{n}]$ for any bounded initial value. Second, when γ_0 slightly larger than $\frac{n+1}{n}$, the global solutions also exist in our numerical tests as shown in Figure 5. So, the upper bound $\frac{n+1}{n}$ of γ_0 may be not the best one. Third, when γ_0 is further larger than the bound $\frac{n+1}{n}$, it is reasonable to conjecture that there is a threshold π^*. There exists a finite time blow-up solution to system (1.1) with the initial value larger than π^* such as Figure 3–8. The Figure 9 and Figure 10 shown that the global solution can also exist if the smaller initial(less than π^*) conditions is proposed, e.g. $u(x,y,0) = 1, u(x,y,0) = (0.5-(x-0.5)^2-(y-0.5)^2), \gamma_0 = 3$, for 2D. It is mean that the existence of global solutions is also depended on the initial states $u(x,0)$ and $w(x,0)$ when γ_0 is further larger than $\frac{n+1}{n}$. The latter two problems are our key problems in theoretical research in the future.

Acknowledgements

This paper is supported by National Natural Science Foundation of China (No.11272277, No. 61304175), the Innovation Scientists and Technicians Troop Construction Projects of Henan Province, China (Grant No. 2017JR0013). Supported by Foundation of Henan Educational Committee (No. 15A110042), Doctoral Scientific research project of Huizhou University (No. C5120201), Program for Young scholar Sponsored by Xuchang University, the key project of Xuchang University (No. 2015109).

References

[1] N. Bellomo, A. Bellouquid, Y. Tao and M. Winkler. Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues. *Mathematical Models and Methods in Applied Sciences*, **25**(9):1663–1763, 2015. https://doi.org/10.1142/S021820251550044X.

[2] X.Y. Chen and W.A. Liu. Global attractor for a density-dependent sensitivity chemotaxis model. *Acta Mathematica Scientia*, **32B**(4):1365–1375, 2012.

[3] X.Y. Chen and Y. Yang. Existence of the traveling wave solutions for a chemotaxis equations. *Acta Mathematica Sinica, Chinese Series*, **55**(5):817–828, 2012.

[4] R. Čiegis and A. Bugajev. Numerical approximation of one model of the bacterial self-organization. *Nonlinear Analysis: Modelling and Control*, **17**(3):253–270, 2012.

[5] D. Horstmann and M. Winkler. Boundedness vs. blow-up in a chemotaxis system. *Journal of Differential Equations*, **215**(1):52–107, 2005. https://doi.org/10.1016/j.jde.2004.10.022.

[6] E.F. Keller and L.A. Segel. Initiation of slime mold aggregation viewed as an instability. *Journal of Theoretical Biology*, **26**(3):399–415, 1970. https://doi.org/10.1016/0022-5193(70)90092-5.

[7] X. Mora. Semilinear parabolic problems define semiflows on c^k spaces. *Transactions of the American Mathematical Society*, **278**(1):21–55, 1983.

[8] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*. Springer-Verlag, New York, 1983. https://doi.org/10.1007/978-1-4612-5561-1.

[9] Y.S. Tao and M. Winkler. Eventual smoothness and stabilization of large-data solutions in a three-dimensional chemotaxis system with consumption of chemoattractant. *Journal of Differential Equations*, **252**(3):2520–2543, 2012. https://doi.org/10.1016/j.jde.2011.07.010.

[10] M. Winkler. Global large-data solutions in a chemotaxis-(Navier-)Stokes system modeling cellular swimming in fluid drops. *Communications in Partial Differential Equations*, **37**(2):319–351, 2012. https://doi.org/10.1080/03605302.2011.591865.

[11] M. Winkler. Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system. *Journal de Mathématiques Pures et Appliqués*, **100**(5):748–767, 2013. https://doi.org/10.1016/j.matpur.2013.01.020.