

ON NONLINEAR SPECTRA FOR SOME NONLOCAL BOUNDARY VALUE PROBLEMS

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Abstract. We consider a second order nonlinear differential equation with nonlocal (integral) condition. The spectrum of it differs essentially from the known ones.

Key words: Fučík problem, Fučík spectrum, nonlocal boundary value problem.

1 Introduction

Investigations of Fučík spectra started in the 1960s. Let us mention the work [3] and the bibliography therein. Of the recent works let us mention [4, 5, 6]. The Fučík spectra have been investigated for the second order equation with different two-points boundary conditions. There are fewer works on the higher order problems.

Our goal is to get formulas for the spectra (λ, μ) of the second order boundary value problem (BVP)

$$x'' = \begin{cases} -(\alpha + 1)\mu^{2\alpha+2}|x|^{2\alpha}x, & \text{if } x \geq 0, \\ -(\alpha + 1)\lambda^{2\alpha+2}|x|^{2\alpha}x, & \text{if } x < 0, \end{cases} \quad (1.1)$$

$(\alpha \geq 0)$ subject to the boundary conditions

$$x(0) = 0, \quad \int_0^1 x(s) ds = 0. \quad (1.2)$$

The spectra is obtained under the normalization condition $|x'(0)| = 1$, because otherwise the problem may have a continuous spectra.

We try to extend investigation of Fučík type spectra in two directions. The first one considers the classical equation with integral boundary condition ([7]). The second direction deals with equations of the type

$$x'' = -\mu f(x^+) + \lambda g(x^-),$$

the good reference for it is the work [6].

This paper is organized as follows. In Section 2 we present results on the Fučík spectrum for the problem $x'' = -\mu^2 x^+ + \lambda^2 x^-$ with the boundary conditions (1.2). In Section 3 we consider problem (1.1) with Dirichlet conditions. In Section 4 we provide formulas for Fučík spectrum of the problem (1.1), (1.2). This is the main result of our work. Our formulas for the spectra are given in terms of the functions $S_\alpha(t)$, which are generalizations of the lemniscatic functions [8] ($\operatorname{sl} t$ and $\operatorname{cl} t$), and their primitives $I_\alpha(t)$. The formulas for relations between lemniscatic functions and their derivatives are known from [2]. The specific case of $\alpha = 1$ is considered in details as an example.

2 Fučík Equation with the Integral Condition

Consider the second order BVP

$$x'' = -\mu^2 x^+ + \lambda^2 x^-, \quad \mu, \lambda > 0, \quad (2.1)$$

where $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$, subject to the boundary conditions

$$x(0) = 0, \quad \int_0^1 x(s) ds = 0. \quad (2.2)$$

DEFINITION 1. The Fučík spectrum is a set of points (λ, μ) such that the problem (2.1), (2.2) has nontrivial solutions.

The first result describes decomposition of the spectrum into branches F_i^+ and F_i^- ($i = 0, 1, 2, \dots$) for the problem (2.1), (2.2).

Proposition 1. *The Fučík spectrum $\Sigma = \bigcup_{i=0}^{+\infty} F_i^\pm$ consists of a set of curves*

$$F_i^+ = \{(\lambda, \mu) \mid x'(0) > 0, \text{ the nontrivial solution } x(t) \text{ of the problem} \\ \text{has exactly } i \text{ zeroes in } (0, 1)\},$$

$$F_i^- = \{(\lambda, \mu) \mid x'(0) < 0, \text{ the nontrivial solution } x(t) \text{ of the problem} \\ \text{has exactly } i \text{ zeroes in } (0, 1)\}.$$

Theorem 1 [[7], section 2]. *The Fučík spectrum $\Sigma = \bigcup_{i=1}^{+\infty} F_i^\pm$ for the problem (2.1), (2.2) consists of the branches given by*

$$F_{2i-1}^+ = \left\{ (\lambda, \mu) \mid \frac{2i\lambda}{\mu} - \frac{(2i-1)\mu}{\lambda} - \frac{\mu \cos(\lambda - \frac{\lambda\pi i}{\mu} + \pi i)}{\lambda} = 0, \right. \\ \left. \frac{i\pi}{\mu} + \frac{(i-1)\pi}{\lambda} \leq 1, \frac{i\pi}{\mu} + \frac{i\pi}{\lambda} > 1 \right\},$$

$$F_{2i}^+ = \left\{ (\lambda, \mu) \mid \frac{(2i+1)\lambda}{\mu} - \frac{2i\mu}{\lambda} - \frac{\lambda \cos(\mu - \frac{\mu\pi i}{\lambda} + \pi i)}{\mu} = 0, \right. \\ \left. \frac{i\pi}{\mu} + \frac{i\pi}{\lambda} \leq 1, \frac{(i+1)\pi}{\mu} + \frac{i\pi}{\lambda} > 1 \right\},$$

$$F_{2i-1}^- = \left\{ (\lambda, \mu) \mid \frac{2i\mu}{\lambda} - \frac{(2i-1)\lambda}{\mu} - \frac{\lambda \cos(\mu - \frac{\mu\pi i}{\lambda} + \pi i)}{\mu} = 0, \right. \\ \left. \frac{(i-1)\pi}{\mu} + \frac{i\pi}{\lambda} \leq 1, \frac{i\pi}{\mu} + \frac{i\pi}{\lambda} > 1 \right\},$$

$$F_{2i}^- = \left\{ (\lambda, \mu) \mid \frac{(2i+1)\mu}{\lambda} - \frac{2i\lambda}{\mu} - \frac{\mu \cos(\lambda - \frac{\lambda\pi i}{\mu} + \pi i)}{\lambda} = 0, \right. \\ \left. \frac{i\pi}{\mu} + \frac{i\pi}{\lambda} \leq 1, \frac{i\pi}{\mu} + \frac{(i+1)\pi}{\lambda} > 1 \right\},$$

where $i = 1, 2, \dots$

Visualization of the spectrum to this problem is given in Figure 1.

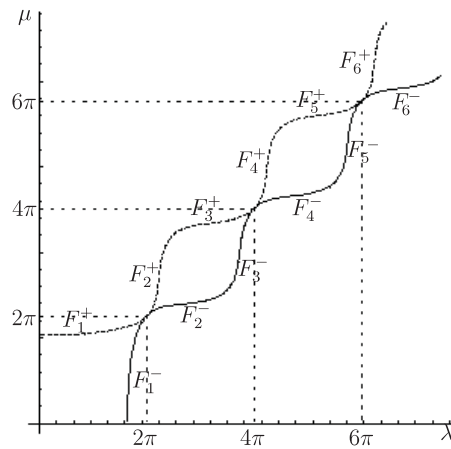


Figure 1. The Fučík spectrum for problem (2.1), (2.2).

3 Spectrum for the Fučík Type Problem with Dirichlet Conditions

Consider the equation

$$x'' = \begin{cases} -(\alpha + 1)\mu^{2\alpha+2}|x|^{2\alpha}x, & \text{if } x \geq 0, \\ -(\alpha + 1)\lambda^{2\alpha+2}|x|^{2\alpha}x, & \text{if } x < 0, \end{cases} \quad (3.1)$$

with the boundary conditions

$$x(0) = x(1) = 0, \quad |x'(0)| = 1, \quad (3.2)$$

where $\alpha \geq 0$, $\lambda, \mu > 0$.

Theorem 2 [[6], subsection 3.2.1]. *The Fučík spectrum $\Sigma = \bigcup_{i=0}^{+\infty} F_i^\pm$ for the problem (3.1), (3.2) consists of the branches given by*

$$\begin{aligned} F_0^+ &= \{(\lambda, 2A_\alpha)\}, & F_0^- &= \{(2A_\alpha, \mu)\}, \\ F_{2i-1}^+ &= \left\{(\lambda, \mu) \mid i\frac{2A_\alpha}{\mu} + i\frac{2A_\alpha}{\lambda} = 1\right\}, \\ F_{2i}^+ &= \left\{(\lambda, \mu) \mid (i+1)\frac{2A_\alpha}{\mu} + i\frac{2A_\alpha}{\lambda} = 1\right\}, \\ F_{2i-1}^- &= \left\{(\lambda, \mu) \mid i\frac{2A_\alpha}{\mu} + i\frac{2A_\alpha}{\lambda} = 1\right\}, \\ F_{2i}^- &= \left\{(\lambda, \mu) \mid i\frac{2A_\alpha}{\mu} + (i+1)\frac{2A_\alpha}{\lambda} = 1\right\}, \end{aligned}$$

where $A_\alpha = \int_0^1 \frac{ds}{\sqrt{1-s^{2\alpha+2}}}$, $i = 1, 2, \dots$

Visualization of the spectrum to this problem is given in Figure 2.

Remark 1. To simplify our formulas we consider equation in the form (3.1), but in the work [6] the authors consider the equation $x'' = -\mu x^+ + \lambda x^-$.

4 Spectrum for the Fučík Type problem with Integral Condition

4.1 Some auxiliary results

The function $S_n(t)$ is defined as a solution of the initial value problem

$$\begin{cases} x'' = -(n+1)x^{2n+1}, \\ x(0) = 0, \quad x'(0) = 1, \end{cases}$$

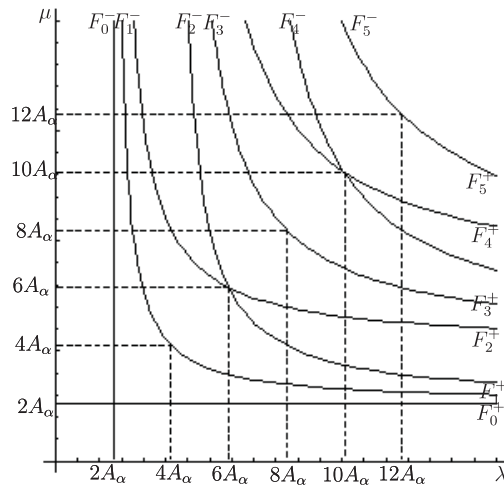


Figure 2. The Fučík spectrum for problem (3.1), (3.2).

where n is a positive integer.

Functions $S_n(t)$ possess some properties of the usual $\sin t$ functions (notice that $S_0(t) = \sin t$). We mention several properties of these functions which are needed in our investigations. The functions $S_n(t)$:

1. are continuous and differentiable;
2. are periodic with the minimal period $4A_n$, where $A_n = \int_0^1 \frac{ds}{\sqrt{1 - s^{2n+2}}}$;
3. take maximal value $+1$ at the points $(4i + 1)A_n$ and minimal value -1 at the points $(4i - 1)A_n$ ($i = 0, \pm 1, \pm 2, \dots$);
4. take zeroes at the points $2iA_n$.

For boundary value problems with the integral condition the following remark may be of value. Let us consider

$$I_n(t) := \int_0^t S_n(\xi) d\xi.$$

This function is periodic with the minimal period $4A_n$ and can be expressed in terms of the so called hypergeometric functions.

Remark 2. A solution of the problem

$$\begin{cases} x'' = -(n + 1)\gamma^{2n+2}|x|^{2n}x, \\ x(0) = 0, \quad x'(0) = 1. \end{cases}$$

can be written in terms of $S_n(t)$ as $x(t) = S_n(\gamma t)/\gamma$.

4.2 The spectrum

Consider the problem

$$x'' = \begin{cases} -(\alpha + 1)\mu^{2\alpha+2}|x|^{2\alpha}x, & \text{if } x \geq 0, \\ -(\alpha + 1)\lambda^{2\alpha+2}|x|^{2\alpha}x, & \text{if } x < 0, \end{cases} \quad (4.1)$$

$$x(0) = 0, \quad \int_0^1 x(s)ds = 0, \quad |x'(0)| = 1, \quad (4.2)$$

where $\alpha \geq 0$, $\lambda, \mu > 0$.

Theorem 3. *The Fučík spectrum $\Sigma = \bigcup_{i=1}^{+\infty} F_i^\pm$ for the problem (4.1), (4.2) consists of the branches given by*

$$F_{2i-1}^+ = \left\{ (\lambda, \mu) \mid i \frac{\lambda}{\mu} I_\alpha(2A_\alpha) - (i-1) \frac{\mu}{\lambda} I_\alpha(2A_\alpha) + \frac{\mu}{\lambda} I_\alpha \left(\lambda - \frac{2i\lambda A_\alpha}{\mu} - 2iA_\alpha \right) = 0, \right. \\ \left. \frac{2A_\alpha}{\mu} i + \frac{2A_\alpha}{\lambda} (i-1) \leq 1, \frac{2A_\alpha}{\mu} i + \frac{2A_\alpha}{\lambda} i > 1 \right\},$$

$$F_{2i}^+ = \left\{ (\lambda, \mu) \mid i \frac{\lambda}{\mu} I_\alpha(2A_\alpha) - i \frac{\mu}{\lambda} I_\alpha(2A_\alpha) + \frac{\lambda}{\mu} I_\alpha \left(\mu - \frac{2i\mu A_\alpha}{\lambda} - 2iA_\alpha \right) = 0, \right. \\ \left. \frac{2A_\alpha}{\mu} i + \frac{2A_\alpha}{\lambda} i \leq 1, \frac{2A_\alpha}{\mu} (i+1) + \frac{2A_\alpha}{\lambda} i > 1 \right\},$$

$$F_{2i-1}^- = \left\{ (\lambda, \mu) \mid i \frac{\mu}{\lambda} I_\alpha(2A_\alpha) - (i-1) \frac{\lambda}{\mu} I_\alpha(2A_\alpha) + \frac{\lambda}{\mu} I_\alpha \left(\mu - \frac{2i\mu A_\alpha}{\lambda} - 2iA_\alpha \right) = 0, \right. \\ \left. \frac{2A_\alpha}{\mu} (i-1) + \frac{2A_\alpha}{\lambda} i \leq 1, \frac{2A_\alpha}{\mu} i + \frac{2A_\alpha}{\lambda} i > 1 \right\},$$

$$F_{2i}^- = \left\{ (\lambda, \mu) \mid i \frac{\mu}{\lambda} I_\alpha(2A_\alpha) - i \frac{\lambda}{\mu} I_\alpha(2A_\alpha) + \frac{\mu}{\lambda} I_\alpha \left(\lambda - \frac{2i\lambda A_\alpha}{\mu} - 2iA_\alpha \right) = 0, \right. \\ \left. \frac{2A_\alpha}{\mu} i + \frac{2A_\alpha}{\lambda} i \leq 1, \frac{2A_\alpha}{\mu} i + \frac{2A_\alpha}{\lambda} (i+1) > 1 \right\},$$

where $i = 1, 2, \dots$

Proof Consider the problem (4.1), (4.2). It is clear that $x(t)$ must have zeroes in $(0, 1)$. That is why $F_0^\pm = \emptyset$. We will prove the theorem for the case of F_{2i-1}^+ . Suppose that $(\lambda, \mu) \in F_{2i-1}^+$ and let $x(t)$ be a respective nontrivial solution of the problem (4.1), (4.2). The solution has $(2i-1)$ zeroes in $(0, 1)$ and $x'(0) = 1$. Let these zeroes be denoted by τ_1, τ_2 and so on.

Consider a solution of problem (4.1), (4.2) in the intervals $(0, \tau_1), (\tau_1, \tau_2), \dots, (\tau_{2i-1}, 1)$. Notice that $|x'(\tau_j)| = 1$ ($j = 1, \dots, 2i-1$). We obtain that problem (4.1), (4.2) in these intervals reduces to the eigenvalue problems. So in the odd numbered intervals we have the problem

$$x'' = -(\alpha + 1)\mu^{2\alpha+2}x^{2\alpha+1}$$

with boundary conditions $x(0) = x(\tau_1) = 0$ in the first such interval and with boundary conditions $x(\tau_{2i-2}) = x(\tau_{2i-1}) = 0$ in the other ones, but in the even intervals we have the problem

$$x'' = -(\alpha + 1)\lambda^{2\alpha+2}x^{2\alpha+1}$$

with boundary condition $x(\tau_{2i-3}) = x(\tau_{2i-2}) = 0$ in each such interval, but for the last one the only condition is $x(\tau_{2i-1}) = 0$. In view of (4.2) the solution $x(t)$ must satisfy the condition

$$\int_0^{\tau_1} x(s) ds + \int_{\tau_2}^{\tau_3} x(s) ds + \dots + \int_{\tau_{2i-2}}^{\tau_{2i-1}} x(s) ds = \left| \int_{\tau_1}^{\tau_2} x(s) ds + \int_{\tau_3}^{\tau_4} x(s) ds + \dots + \int_{\tau_{2i-1}}^1 x(s) ds \right|. \quad (4.3)$$

Since $x(t) = S_\alpha(\mu t)$ in the interval $(0, \tau_1)$ and $x(\tau_1) = 0$ we obtain $\tau_1 = \frac{2A_\alpha}{\mu}$. Analogously we obtain for the other zeroes

$$\begin{aligned} \tau_2 &= \frac{2A_\alpha}{\mu} + \frac{2A_\alpha}{\lambda}, & \tau_3 &= 2\frac{2A_\alpha}{\mu} + \frac{2A_\alpha}{\lambda}, \\ &\dots\dots\dots & & \\ \tau_{2i-2} &= (i-1)\frac{2A_\alpha}{\mu} + (i-1)\frac{2A_\alpha}{\lambda}, \\ \tau_{2i-1} &= i\frac{2A_\alpha}{\mu} + (i-1)\frac{2A_\alpha}{\lambda}. \end{aligned}$$

In view of these facts it is easy to get that

$$\int_0^{\tau_1} x(s) ds = \int_{\tau_2}^{\tau_3} x(s) ds = \int_{\tau_4}^{\tau_5} x(s) ds = \dots = \int_{\tau_{2i-2}}^{\tau_{2i-1}} x(s) ds = \frac{1}{\mu^2} I_\alpha(2A_\alpha).$$

Therefore

$$\int_0^{\tau_1} x(s) ds + \int_{\tau_2}^{\tau_3} x(s) ds + \dots + \int_{\tau_{2i-2}}^{\tau_{2i-1}} x(s) ds = i\frac{1}{\mu^2} I_\alpha(2A_\alpha).$$

Now we consider a solution of the problem (4.1), (4.2) in the remaining intervals. Since $x(t) = -S_\alpha(\lambda t - \lambda\tau_1)$ in (τ_1, τ_2) we obtain

$$\int_{\tau_1}^{\tau_2} x(s) ds = \int_{\tau_3}^{\tau_4} x(s) ds = \int_{\tau_5}^{\tau_6} x(s) ds = \dots = \int_{\tau_{2i-3}}^{\tau_{2i-2}} x(s) ds = -\frac{1}{\lambda^2} I_\alpha(2A_\alpha).$$

But in the last interval $(\tau_{2i-1}, 1)$ we obtain

$$\int_{\tau_{2i-1}}^1 x(s) ds = \frac{1}{\lambda^2} I_\alpha \left(\lambda - \lambda \frac{2A_\alpha}{\mu} i - 2A_\alpha i \right).$$

It follows from the last two lines that

$$\begin{aligned} \left| \int_{\tau_1}^{\tau_2} x(s) ds + \int_{\tau_3}^{\tau_4} x(s) ds + \dots + \int_{\tau_{2i-3}}^{\tau_{2i-2}} x(s) ds + \int_{\tau_{2i-1}}^1 x(s) ds \right| \\ = (i-1) \frac{1}{\lambda^2} I_\alpha(2A_\alpha) - \frac{1}{\lambda^2} I_\alpha \left(\lambda - \lambda \frac{2A_\alpha}{\mu} i - 2A_\alpha i \right). \end{aligned}$$

In view of this equality and (4.3) we obtain

$$i \frac{1}{\mu^2} I_\alpha(2A_\alpha) = (i-1) \frac{1}{\lambda^2} I_\alpha(2A_\alpha) - \frac{1}{\lambda^2} I_\alpha \left(\lambda - \lambda \frac{2A_\alpha}{\mu} i - 2A_\alpha i \right).$$

Multiplying it by $\mu\lambda$, we obtain

$$i \frac{\lambda}{\mu} I_\alpha(2A_\alpha) - (i-1) \frac{\mu}{\lambda} I_\alpha(2A_\alpha) + \frac{\mu}{\lambda} I_\alpha \left(\lambda - \frac{2i\lambda A_\alpha}{\mu} - 2iA_\alpha \right) = 0. \quad (4.4)$$

Considering the solution of problem (4.1), (4.2) it is easy to prove that

$$\tau_{2i-1} \leq 1 < \tau_{2i} \quad \text{or} \quad \frac{2A_\alpha}{\mu} i + \frac{2A_\alpha}{\lambda} (i-1) \leq 1 < \frac{2A_\alpha}{\mu} i + \frac{2A_\alpha}{\lambda} i.$$

This result and (4.4) prove the theorem for the case of F_{2i-1}^+ . The proof for other branches is analogous. ■

Remark 3. If $\alpha = 0$ we obtain problem (2.1), (2.2). The spectrum of this problem is given in Figure 1.

4.3 The example for $\alpha = 1$

Now we consider the problem (4.1), (4.2) for the case of $\alpha = 1$. It can be written as

$$\begin{cases} x'' = -2\mu^4 x^{3+} + 2\lambda^4 x^{3-}, & \mu, \lambda \geq 0, \\ x(0) = 0, \quad \int_0^1 x(s) ds = 0, & |x'(0)| = 1. \end{cases} \quad (4.5)$$

where $x^{3+} = \max\{x^3, 0\}$, $x^{3-} = \max\{-x^3, 0\}$.

Theorem 4. *The Fučík spectrum of the problem (4.5) consists of the branches given by*

$$F_{2i-1}^+ = \left\{ (\lambda, \mu) \mid \frac{2i\lambda\pi}{\mu} - \frac{(2i-1)\mu\pi}{\lambda} - \frac{\mu \arctan \operatorname{cl}(\lambda - \lambda \frac{2A}{\mu} i - 2Ai)}{\lambda} = 0, \right. \\ \left. i \frac{2A}{\mu} + (i-1) \frac{2A}{\lambda} \leq 1, i \frac{2A}{\mu} + i \frac{2A}{\lambda} > 1 \right\},$$

$$F_{2i}^+ = \left\{ (\lambda, \mu) \mid \frac{(2i+1)\lambda\pi}{\mu} - \frac{2i\mu\pi}{\lambda} - \frac{\lambda \arctan \operatorname{cl}(\mu - \mu \frac{2A}{\lambda} i - 2Ai)}{\mu} = 0, \right. \\ \left. i \frac{2A}{\mu} + i \frac{2A}{\lambda} \leq 1, (i+1) \frac{2A}{\mu} + i \frac{2A}{\lambda} > 1 \right\},$$

$$F_{2i-1}^- = \left\{ (\lambda, \mu) \mid \frac{2i\mu\pi}{\lambda} - \frac{(2i-1)\lambda\pi}{\mu} - \frac{\lambda \arctan \operatorname{cl}(\mu - \mu \frac{2A}{\lambda} i - 2Ai)}{\mu} = 0, \right. \\ \left. (i-1) \frac{2A}{\mu} + i \frac{2A}{\lambda} \leq 1, i \frac{2A}{\mu} + i \frac{2A}{\lambda} < 1 \right\},$$

$$F_{2i}^- = \left\{ (\lambda, \mu) \mid \frac{(2i+1)\mu\pi}{\lambda} - \frac{2i\lambda\pi}{\mu} - \frac{\mu \arctan \operatorname{cl}(\lambda - \lambda \frac{2A}{\mu} i - 2Ai)}{\lambda} = 0, \right. \\ \left. i \frac{2A}{\mu} + i \frac{2A}{\lambda} \leq 1, i \frac{2A}{\mu} + (i+1) \frac{2A}{\lambda} > 1 \right\},$$

where $\operatorname{cl}(t)$ is the lemniscatic cosine function, $A = \int_0^1 \frac{ds}{\sqrt{1-s^4}}$, $i = 1, 2, \dots$

Proof We will prove this theorem only for F_{2i-1}^+ . The proof for other branches is analogous. It is well-known that $S_1(t) = \operatorname{sl}(t)$, where $\operatorname{sl}t$ is the lemniscatic sine function. It is known (see, [1]) that

$$\int_0^t \operatorname{sl} s \, ds = \frac{\pi}{4} - \arctan \operatorname{cl} t.$$

Thus we obtain

$$I_1(2A) = \frac{\pi}{4} - \arctan \operatorname{cl} 2A = \frac{\pi}{4} - \arctan(-1) = 2\frac{\pi}{4},$$

$$I_1\left(\lambda - \frac{2i\lambda A}{\mu} - 2iA\right) = \frac{\pi}{4} - \arctan \operatorname{cl}\left(\lambda - \frac{2i\lambda A}{\mu} - 2iA\right).$$

Using these expressions, equation from Theorem 3 can be written as

$$\begin{aligned}
 & i\frac{\lambda}{\mu}I_1(2A) - (i-1)\frac{\mu}{\lambda}I_1(2A) + \frac{\mu}{\lambda}I_2\left(\lambda - \frac{2i\lambda A}{\mu} - 2iA\right) \\
 &= 2i\frac{\lambda}{\mu}\frac{\pi}{4} - 2(i-1)\frac{\mu}{\lambda}\frac{\pi}{4} + \frac{\mu}{\lambda}\left(\frac{\pi}{4} - \arctan \operatorname{cl}\left(\lambda - \frac{2i\lambda A}{\mu} - 2iA\right)\right) \\
 &= \frac{2i\lambda}{\mu}\frac{\pi}{4} - \frac{(2i-1)\mu}{\lambda}\frac{\pi}{4} - \frac{\mu}{\lambda}\arctan \operatorname{cl}\left(\lambda - \frac{2i\lambda A}{\mu} - 2iA\right) = 0.
 \end{aligned}$$

■

Visualization of the spectrum to this problem is given in Figure 3.

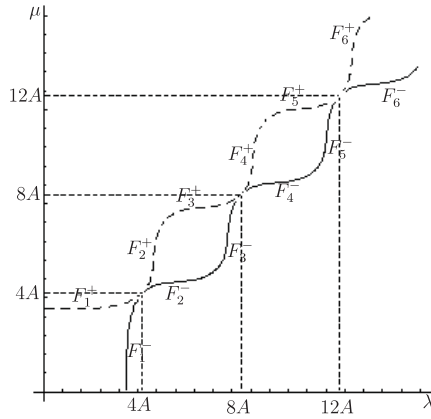


Figure 3. The Fučík spectrum for problem (4.5).

Remark 4. If $\alpha = 0$, then we obtain problem (2.1), (2.2). The spectrum of this problem is given in Figure 1. These spectra are structurally identical.

Remark 5. Let us mention also that the proof of Theorem 4 may be conducted in the same way as for Theorem 3.

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