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Spectral Approximation Methods for Fredholm Integral Equations with Non-Smooth Kernels

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Abstract. In this paper, polynomially based projection and modified projection methods for approximating the solution of Fredholm integral equations with a kernel of Green's function type are studied. The projection is either an orthogonal projection or an interpolatory projection using Legendre polynomial basis. The orders of convergence of these methods and those of superconvergence of the iterated versions are analysed. A numerical example is given to illustrate the theoretical results.

Keywords: Fredholm integral equation, orthogonal projection, interpolatory projection, Legendre polynomial, superconvergence.

AMS Subject Classification: 45Gxx; 47H30.

1 Introduction

Consider the Fredholm integral equation defined on $\mathbb{X} = L^\infty[-1,1]$ by

$$u(s) - \int_{-1}^{1} \kappa(s,t)u(t)dt = f(s), \quad s \in [-1,1], \quad u \in \mathbb{X}, \tag{1.1}$$

where κ is a real-valued function. Assume that for $f \in \mathbb{X}$ the above equation has a unique solution u. The projection methods are standard methods for finding numerical solutions of equations of type (1.1) (see Atkinson [3]). These methods include the Galerkin method based on the orthogonal projection and the collocation method based on an interpolatory projection. Sloan in [18] introduced iterated Galerkin/iterated collocation solutions obtained by one step of iteration which improves upon the projection solution. In Kulkarni [12] a new method (so-called modified projection or multi-projection method) based on projections is proposed for solving (1.1). It is shown that if κ and f are suitably smooth, the resulting solution converges faster than the projection and the Sloan solutions. Moreover, the iterated modified projection solution obtained by performing one step of iteration converges the fastest of all. These results have been extended in Kulkarni [13] to the case when the kernel fails to be sufficiently differentiable because of discontinuities along the diagonal. For this kind of kernels, convergence results for the projection methods have been already discussed by Chatelin and Lebbar in [7, 8], while the discrete Galerkin method and its iterated version were discussed in Atkinson and Potra [5].

It is well known that to get better precision in these methods by using piecewise polynomial approximation, the number of partition points should be increased. Hence, in such cases, we should solve a large system of linear equations, which is computationally expensive.

This paper aims to investigate the projection and the modified projection methods to solve Equation (1.1) with a non-smooth kernel using global polynomial basis functions rather than piecewise polynomial basis functions which reduces highly the size of linear system. In fact, integral equations with Green's function type kernel are of interest and seem not to be studied at the moment by global polynomials. In particular, Legendre polynomials can be used as basis functions which have nice properties of orthogonality and low computational cost. Obviously, low degree polynomials imply small linear systems, something which is highly appreciated in practical computations. We obtain improved order bounds as compared to the existing methods. Indeed, we show that the iterated modified projection method performs better than the projection, Sloan and modified projection methods. Note that, the size of the system of equations in the Legendre modified projection methods remains the same as in the projection methods. For this reason, this method has benefits theoretically and computationally.

In several recent papers, various polynomially based numerical methods for linear integral equations were studied. In [3, Chap.3] the author considers the use of trigonometric polynomials as approximations in solving Fredholm integral equations with periodic kernels. The discrete Galerkin method using Legendre polynomials was introduced in Golberg [10] and was extended by the same author in [9] to some equations with singular kernels. The discrete iterated Galerkin method was proposed in Kulkarni and Gnaneshwar [14] and the convergence of the Legendre-Galerkin solution in the case of weakly singular kernels was considered in Panigrahi and Gnaneshwar [16]. Moreover, the Legendre multi-projection, as well as its iterated version, were studied in G. Long et al. [15]. In Allouch et al. [2], Legendre superconvergent projection-type methods for nonlinear Hammerstein equations were proposed.

Here is an outline of the paper. In Section 2, the notations are set and some preliminary results are recalled. In Section 3, the orders of convergence of the proposed methods and those of their iterated versions for both the orthogonal and the interpolatory projection are obtained. Numerical results are given in Section 4.

2 Method and notations

Let $\Pi = [-1,1] \times [-1,1]$. Divide Π into two parts Π_1 and Π_2, where

$$\Pi_1 = \{(s,t) : -1 \le s \le t \le 1\} \quad \text{and} \quad \Pi_2 = \{(s,t) : -1 \le t \le s \le 1\}.$$

Let α and γ be two integers such that $\alpha \ge \gamma$, $\alpha \ge 0$ and $\gamma \ge -1$. We assume that the kernel κ defined in (1.1) has the following form

$$\kappa(s,t) = \begin{cases} \kappa_1(s,t), & (s,t) \in \Pi_1, \quad s \neq t, \\ \kappa_2(s,t), & (s,t) \in \Pi_2, \end{cases}$$

where $\kappa_i \in \mathcal{C}^{\alpha}(\Pi_i)$, $i = 1,2$. If $\gamma \ge 0$, then $\kappa \in \mathcal{C}^{\gamma}(\Pi)$. If $\gamma = -1$, then κ may have a discontinuity of the first kind along the line $s = t$. Following Chatelin and Lebbar [8], we say that κ is of class $\mathcal{C}(\alpha,\gamma)$. For $\mu = 0,\ldots,\alpha$, set

$$M_{i,\mu} = \max\left\{\left|\frac{\partial^{\mu}\kappa_i}{\partial t^{\mu}}(s,t)\right| : (s,t) \in \Pi_i\right\},\ i=1,2, \quad M_{\mu} = \max\{M_{1,\mu}, M_{2,\mu}\}.$$

Let T be the integral operator defined by

$$(Tx)(s) = \int_{-1}^{1} \kappa(s,t)x(t)dt, \quad s \in [-1,1].$$

The operator T is compact and is completely continuous from $L^{\infty}[-1,1]$ into $\mathcal{C}^{\gamma_1}[-1,1]$, where $\gamma_1 = \min\{\alpha, \gamma+1\}$.

Equation (1.1) can be written symbolically as

$$u - Tu = f. \tag{2.1}$$

Let u be the unique solution of (2.1). If $f \in \mathcal{C}^{\alpha}[-1,1]$, then from Corollary 3.2 of Atkinson and Potra [4], $u \in \mathcal{C}^{\alpha}[-1,1]$. If $x \in \mathcal{C}[-1,1]$, then

$$(Tx)^{(\mu)}(s) = \int_{0}^{1} \frac{\partial^{\mu}\kappa}{\partial s^{\mu}}(s,t)x(t)dt, \quad 0 \le \mu \le \gamma_1, \tag{2.2}$$

where the kernel $q(s,t) = \frac{\partial^{\mu}\kappa}{\partial s^{\mu}}(s,t) \in \mathcal{C}(\alpha-\mu, \gamma-\mu)$. According to Kulkarni [13], we have

$$\left\|(Tx)^{(\mu)}\right\|_{\infty} \le c\,\|x\|_{\infty}, \quad 0 \le \mu \le \gamma_1 + 1. \tag{2.3}$$

Let $\mathbb{X}_n$ be the set of all polynomials of degree $\le n$ defined on $[-1,1]$. Then the dimension of $\mathbb{X}_n$ is $n+1$, and the Legendre polynomials $\{L_0, L_1, L_2, \ldots, L_n\}$ defined by

$$L_0(s) = 1, \quad L_1(s) = s, \quad s \in [-1,1],$$
$$(i+1)L_{i+1}(s) = (2i+1)sL_i(s) - iL_{i-1}(s), \quad i = 1,2,\ldots,n-1$$

form an orthogonal basis for $\mathbb{X}_n$. Since

$$\langle L_i, L_j \rangle = \begin{cases} 2/(2i+1), & i = j, \\ 0, & i \neq j, \end{cases}$$

an orthonormal basis for $\mathbb{X}_n$ is given by $\left\{\varphi_i(s) = \sqrt{\frac{2i+1}{2}} L_i(s) : i = 0, 1, \dots n\right\}$.

Orthogonal projection operator. For $x, y \in L^2[-1,1]$, the inner product is given by

$$\langle x, y \rangle = \int_{-1}^{1} x(t)y(t)dt \quad \text{and norm is} \quad \|x\|_{L^2} = \left(\int_{-1}^{1} x(t)^2 dt\right)^{\frac{1}{2}}.$$

Let $\pi_n^G u$ denote the restriction to $\mathbb{X}$ of the orthogonal projection from $L^2[-1,1]$ onto $\mathbb{X}_n$. Then for $x \in \mathbb{X}$

$$\langle \pi_n^G x, \varphi_i \rangle = \langle x, \varphi_i \rangle, \quad i = 0, 1, \dots, n. \tag{2.4}$$

Interpolatory projection operator. For $x \in \mathcal{C}[-1,1]$, let $\pi_n^C x$ denote the unique polynomial of degree n that satisfies

$$(\pi_n^C x)(\tau_i) = x(\tau_i), \quad i = 0, 1, \dots, n, \tag{2.5}$$

where $\{\tau_0, \tau_1, \dots, \tau_n\}$ are the zeros of the Legendre polynomial L_{n+1}. Clearly, π_n^C is a linear projection operator on $\mathcal{C}[-1,1]$, whose range is $\mathbb{X}_n$. For notational convenience, from now on we write π_n^G or π_n^C as π_n. Throughout this paper C will denotes a generic constant independent of n.

Lemma 1. *(Jackson's theorem). Let $r \geq 0$ be an integer. For any $x \in \mathcal{C}^r[-1,1]$,*

$$\inf_{v \in \mathbb{X}_n} \|x - v\|_{L^2} \leq C n^{-r} \|x^{(r)}\|_{L^2},$$

where C is independent of x (Schumaker [17], page 96).

We define $H^r[-1,1]$ to be the Hilbert space of the functions $x \in L^2[-1,1]$ such that all the derivatives of x of order up to r can be represented by functions in $L^2[-1,1]$. In short,

$$H^r[-1,1] = \left\{x \in L^2[-1,1] : x^{(k)} \in L^2[-1,1],\, 0 \leq k \leq r\right\}.$$

The crucial properties of π_n are given in the following lemma.

Lemma 2. *(Canuto et al. [6, p.287]). Let $\pi_n : \mathcal{C}[-1,1] \to \mathbb{X}_n$ be the projection operator defined by (2.4) or (2.5). There exists a constant $p > 0$ independent of n such that for any $n \in \mathbb{N}$ and any $x \in L^2[-1,1]$,*

$$\|\pi_n x\|_{L^2} \leq p\|x\|_{L^2}, \tag{2.6}$$

$$\|x - \pi_n x\|_{L^2} \leq (1 + p) \inf_{v \in \mathbb{X}_n} \|x - v\|_{L^2} \to 0,\ n \to \infty. \tag{2.7}$$

Whenever $x \in H^r[-1,1]$ *with* $r \geq 1$, *one has*

$$\|x - \pi_n x\|_{L^2} \leq C_1 n^{-r} \|x^{(r)}\|_{L^2}, \tag{2.8}$$

$$\|x - \pi_n x\|_\infty \leq C_1 n^{\beta - r} \|x^{(r)}\|_{L^2}, \quad n \geq r-1, \tag{2.9}$$

where C_1 *is a constant independent of* n, $\beta = \frac{3}{4}$ *for the orthogonal projection and* $\beta = \frac{1}{2}$ *for the interpolatory projection.*

The estimate (2.7) shows that $\|x - \pi_n x\|_{L^2} \to 0$ as $n \to \infty$ for all $x \in \mathcal{C}[-1,1]$, whereas the estimate (2.9) imply that $\|x - \pi_n x\|_\infty \to 0$ as $n \to \infty$, for any $x \in \mathcal{C}^r[-1,1]$ with $r \geq 1$.

The operator π_n is unbounded in the infinity norm. More precisely,

$$\|\pi_n^G\|_\infty \leq C \log n, \tag{2.10}$$

(see Golberg [10, 11]) and

$$\|\pi_n^C\|_\infty = 1 + \frac{2^{\frac{3}{2}}}{\sqrt{\pi}} n^{\frac{1}{2}} + B_0 + \mathcal{O}(n^{-\frac{1}{2}}), \tag{2.11}$$

where B_0 is a bounded constant (see Tang et al. [19]).

Note that, using (2.2) and (2.3), we see that T is a continuous operator on $L^2[-1,1]$ into $H^{\gamma_1+1}[-1,1]$. On the one hand

$$\|(Tx)^{(\mu)}\|_{L^2} \leq \max_{s \in [-1,1]} \left(\int_{-1}^{1} \left| \frac{\partial^\mu \kappa}{\partial s^\mu}(s,t) \right|^2 dt \right)^{\frac{1}{2}} \|x\|_{L^2}, \quad 0 \leq \mu \leq \gamma_1.$$

On the other hand, if x is a piecewise continuous function over a quasi-uniform partition Δ of $[-1,1]$, then for $s \notin \Delta$ (see Kulkarni [13])

$$(Tx)^{(\gamma_1+1)}(s) = \int_{-1}^{1} \frac{\partial^{\gamma_1+1}}{\partial s^{\gamma_1+1}} \kappa(s,t) x(t) dt + \frac{\partial^{\gamma_1}}{\partial s^{\gamma_1}} \kappa_1(s,s) x(s) - \frac{\partial^{\gamma_1}}{\partial s^{\gamma_1}} \kappa_2(s,s) x(s) \tag{2.12}$$

and the values $(Tx)^{(\gamma_1+1)}(s+0)$ and $(Tx)^{(\gamma_1+1)}(s-0)$ will exist using limits in (2.12), for all $s \in \Delta$. Hence, we deduce that $\|(Tx)^{(\gamma_1+1)}\|_{L^2}$ can be bounded by $C\|x\|_{L^2}$. Finally, we conclude that for each $x \in L^2[-1,1]$,

$$\|(Tx)^{(\mu)}\|_{L^2} \leq K \|x\|_{L^2}, \quad 0 \leq \mu \leq \gamma_1 + 1, \tag{2.13}$$

where K is a positive constant depending on κ.

In the classical projection method, Equation (1.1) is approximated by

$$u_n - \pi_n T u_n = \pi_n f,$$

while in the iterated projection method proposed by Sloan, it is approximated by

$$\widetilde{u}_n - T \pi_n \widetilde{u}_n = f.$$

In order to obtain a more accurate approximation solution than $\widetilde{u}_n$, the following modified projection method is proposed in Kulkarni [12]

$$u_n^M - T_n^M u_n^M = f, \tag{2.14}$$

where

$$T_n^M = \pi_n T + T\pi_n - \pi_n T\pi_n.$$

The iterated modified projection solution is defined as

$$\widetilde{u}_n^M = Tu_n^M + f$$

and it may converge to u at a faster rate than the approximation u_n^M does. As for the projection methods, Equation (2.14) can be reduced to a system of linear equations of size $n+1$. To discuss the existence and uniqueness of the approximate solutions we need first to recall the following definition of ν-convergence and a lemma from [1].

Definition 1. (ν-convergence) Let $\mathbb{X}$ be Banach space and $BL(\mathbb{X})$ be space of bounded linear operators from $\mathbb{X}$ into $\mathbb{X}$. Let $A, A_n \in BL(\mathbb{X})$. We say that A_n is ν–convergent to A if

$$(H_1)\ \ \|A_n\| \le c < \infty, \quad (H_2)\ \ \|(A_n - A)A\| \to 0 \text{ as } n \to \infty,$$
$$(H_3)\ \ \|(A_n - A)A_n\| \to 0 \text{ as } n \to \infty.$$

Lemma 3. *(Ahues et al. [1]) Let $\mathbb{X}$ be a Banach space and A, A_n be bounded linear operators on $\mathbb{X}$. If $\|A_n - A\| \to 0$, as $n \to \infty$ or A_n is ν-convergent to A and $(I - A)^{-1}$ exists, then for n large enough $(I - A_n)^{-1}$ exists and is uniformly bounded on $\mathbb{X}$.*

3 Convergence rates

Lemma 4. *Assume that the inverse of $(I-T)$ exists and is uniformly bounded. Then, for a sufficiently large n, the operators $(I - \pi_n T)^{-1}$ and $(I - T\pi_n)^{-1}$ exist. Moreover,*

$$\|(I - \pi_n T)^{-1}\|_\infty \le C_1 \quad \text{and} \quad \|(I - T\pi_n)^{-1}\|_\infty \le C_2$$

for a suitable constants C_1 and C_2 independents of n.

Proof. We need to show that $T\pi_n$ is ν-convergent to T. Let $x \in \mathcal{C}[-1,1]$ and let $s \in [-1,1]$. By using the Cauchy-Schwarz inequality, one has

$$|(T\pi_n x)(s)| = \left|\int_{-1}^{1} \kappa(s,t)(\pi_n x)(t)dt\right| \le \left[\int_{-1}^{1} |\kappa(s,t)|^2 dt\right]^{\frac{1}{2}} \times \left[\int_{-1}^{1} |(\pi_n x)(t)|^2 dt\right]^{\frac{1}{2}}.$$

Hence, by (2.6)

$$\|T\pi_n x\|_\infty \le Ap\|x\|_{L^2} \le \sqrt{2}Ap\|x\|_\infty, \tag{3.1}$$

where

$$A = \max_{s\in[-1,1]} \left[\int_{-1}^{1} |\kappa(s,t)|^2 dt\right]^{\frac{1}{2}}.$$

Thus, (H_1) is satisfied with $c = \sqrt{2}Ap$. Similarly to (3.1), we have

$$\|T(I-\pi_n)Tx\|_\infty \leq A\|(I-\pi_n)Tx\|_{L^2} \leq \sqrt{2}A\|(I-\pi_n)T\|_{L^2}\|x\|_\infty.$$

Since T is a compact linear integral operator in $L^2[-1,1]$ and π_n converges to the identity operator pointwise, then it follows that $\|(I-\pi_n)T\|_{L^2} \to 0$ as $n \to \infty$. Hence, we deduce that

$$\|T(I-\pi_n)T\|_\infty \leq \sqrt{2}A\|(I-\pi_n)T\|_{L^2} \to 0 \text{ as } n \to \infty.$$

This proves (H_2). The condition (H_3) can be checked in the same way as (H_2). According to Lemma 3, the operators $(I - T\pi_n)^{-1}$ exist and are uniformly bounded for all sufficiently large n. The existence and the uniform boundedness of the operators $(I-\pi_n T)^{-1}$ follows immediately from `Lemma 3.4.1` of Atkinson [3]. This completes the proof. □

The following well known error estimates are quoted from Atkinson [3, Chap.3].

$$\|u-u_n\|_\infty \leq \|(I-\pi_n T)^{-1}\|_\infty\|(I-\pi_n)u\|_\infty, \tag{3.2}$$

$$\|u-\widetilde{u}_n\|_\infty \leq \|(I-T\pi_n)^{-1}\|_\infty\|T(I-\pi_n)u\|_\infty. \tag{3.3}$$

For a fixed $s \in [-1,1]$, let $\kappa_s(t) \equiv \kappa(s,t)$ for $t \in [-1,1]$ be the s section of κ. The function κ_s, is of class $\mathcal{C}^{\gamma_1}$ in $[-1,s]$ and $[s,1]$. Furthermore, by our assumptions on κ_1 and κ_2, the values $\kappa_s^{(\gamma_1)}(s-0)$ and $\kappa_s^{(\gamma_1)}(s+0)$ will exist. Then, $\kappa_s^{(\gamma_1)}$ is bounded in this way

$$\|\kappa_s^{(\gamma_1)}\|_\infty \leq 2M_{\gamma_1}. \tag{3.4}$$

Corollary 1. Let the kernel κ be of class $\mathcal{C}(\alpha,\gamma)$. Then for each $s \in [-1,1]$, there exists a polynomial v_s of degree $\leq n$ such that

$$\|\kappa_s^{(\gamma_1)} - v_s\|_{L^2} = \mathcal{O}(n^{-\frac{1}{2}(\gamma_2-\gamma_1)}), \tag{3.5}$$

where $\gamma_2 = \min\{\alpha, \gamma+2\}$.

Proof. Let

$$\ell(s,t) = \frac{\partial^{\gamma_1}\kappa}{\partial t^{\gamma_1}}(s,t), \quad s,t \in [-1,1].$$

For a fixed $s \in [-1,1]$, we denote $\ell_s(t) = \ell(s,t)$, $t \in [-1,1]$. Since $\ell_s \in L^2[-1,1]$, for all $s \in [-1,1]$ from Schumaker [17, p.92] there is a polynomial v_s of degree $\leq n$ such that

$$\|\ell_s - v_s\|_{L^2} \leq C_1\omega\big(\ell_s, 1/n\big),$$

where ω denotes the modulus of smoothness of ℓ_s with respect to the L^2- norm

$$\omega\Big(\ell_s, \frac{1}{n}\Big) = \sup_{0\leq\delta\leq\frac{1}{n}} \left(\int_{-1}^{1-\delta} |\ell_s(t+\delta)-\ell_s(t)|^2 dt\right)^{\frac{1}{2}}. \tag{3.6}$$

Note that if $\alpha \le \gamma_1$, we have $\gamma_2 - \gamma_1 = 0$ and (3.5) holds, since

$$\|\kappa_s^{(\gamma_1)} - v_s\|_{L^2} \le 2\sqrt{2}M_{\gamma_1} + \max_{s\in[-1,1]} \|v_s\|_{L^2} = \mathcal{O}(1).$$

Suppose $\alpha \ge \gamma_1 + 1$, and let $|s| \le 1 - 3\delta/2$. Then

$$\begin{aligned}\int_{-1}^{1-\delta} |\ell_s(t+\delta) - \ell_s(t)|^2 dt &= \int_{-1}^{s-\frac{3\delta}{2}} |\ell_{2s}(t+\delta) - \ell_{2s}(t)|^2 dt \\ &+ \int_{s-\frac{3\delta}{2}}^{s-\frac{\delta}{2}} |\ell_s(t+\delta) - \ell_{2s}(t)|^2 dt + \int_{s-\frac{\delta}{2}}^{s+\frac{\delta}{2}} |\ell_{1s}(t+\delta) - \ell_s(t)|^2 dt \\ &+ \int_{s+\frac{\delta}{2}}^{1-\delta} |\ell_{1s}(t+\delta) - \ell_{1s}(t)|^2 dt,\end{aligned} \tag{3.7}$$

where $\ell_{is} = \kappa_{is}^{(\gamma_1)}$ and κ_{is} is the s section of κ_i, $i = 1, 2$. The function $\ell_{is} \in \mathcal{C}^{\alpha-\gamma_1}(\Pi_i)$, then using the mean value theorem,

$$|\ell_{is}(t+\delta) - \ell_{is}(t)| \le \sup_{-1\le s,t\le 1} \left| \frac{\partial^{\gamma_1+1}\kappa_i}{\partial t^{\gamma_1+1}}(s,t) \right| \delta \le M_{i,\gamma_1+1}\delta.$$

The second and the third integral in (3.7) can be bounded by $4(M_{2,\gamma_1})^2\delta$. Thus, the resulting global bound for (3.6) is $\mathcal{O}(\delta^{\frac{1}{2}})$. In the case $|s| \ge 1 - \frac{3\delta}{2}$, the proof is essentially the same and this reach the proof. □

Theorem 1. *Let the kernel κ be of class $\mathcal{C}(\alpha, \gamma)$. Then for each $s \in [-1, 1]$*

$$\max_{s\in[-1,1]} \|\kappa_s - \pi_n\kappa_s\|_{L^2} = \mathcal{O}(n^{-\frac{1}{2}(\gamma_1+\gamma_2)}). \tag{3.8}$$

Proof. Note that if $s = -1$,

$$\|\kappa_s - \pi_n\kappa_s\|_{L^2} = \|\kappa_{2s} - \pi_n\kappa_{2s}\|_{L^2} = \mathcal{O}(n^{-\alpha})$$

and a similar estimate is obtained for $s = 1$. Thus, (3.8) holds. From (3.4), κ_s belongs to $H^{\gamma_1}[-1, 1]$, and the estimate (2.8) yields

$$\|\kappa_s - \pi_n\kappa_s\|_{L^2} \le C_1 n^{-\gamma_1} \|\kappa_s^{(\gamma_1)}\|_{L^2}. \tag{3.9}$$

We observe that replacing $\kappa^{(\gamma_1)}$ by $\kappa^{(\gamma_1)} - v_s$ leaves the left-hand side of (3.9) unchanged since $\pi_n v_s = v_s$. Hence, the estimate (3.5) concludes the proof. □

Lemma 5. *Let T be an integral operator with a kernel $\kappa \in \mathcal{C}(\alpha, \gamma)$. Then the operators $(I - T_n^M)^{-1}$ exist and are uniformly bounded for a sufficiently large n, i.e. there exists a constant $C_3 > 0$ such that*

$$\|(I - T_n^M)^{-1}\|_\infty \le C_3 < \infty. \tag{3.10}$$

Proof. Let π_n^G be the orthogonal projection defined by (2.4). We first note that from (2.10) it follows that

$$\|(T - T_n^M)x\|_\infty = \|(I - \pi_n^G)T(I - \pi_n^G)x\|_\infty \le (1 + C\log n)\|T(I - \pi_n^G)x\|_\infty.$$

Since

$$\langle v, (I - \pi_n^G)x\rangle = 0, \quad \forall v \in \mathbb{X}_n,$$

we can write

$$\begin{aligned} |[T(I - \pi_n^G)x](s)| &= \left|\int_{-1}^{1} \kappa(s,t)(x - \pi_n^G x)(t)dt\right| \\ &= |\langle \kappa_s, (I - \pi_n^G)x\rangle| = |\langle (I - \pi_n^G)\kappa_s, (I - \pi_n^G)x\rangle|. \end{aligned} \tag{3.11}$$

It results now from the Cauchy-Schwarz inequality that

$$\|(T - T_n^M)x\|_\infty \leq (1 + C\log n) \max_{s\in[-1,1]} \|(I - \pi_n^G)\kappa_s\|_{L^2}\|(I - \pi_n^G)x\|_{L^2}.$$

Thus,

$$\|T - T_n^M\|_\infty \leq \sqrt{2}(1 + p)(1 + C\log n) \max_{s\in[-1,1]} \|(I - \pi_n^G)\kappa_s\|_{L^2}.$$

Therefore, estimate (3.8) leads to

$$\|T - T_n^M\|_\infty = \mathcal{O}(n^{-\frac{1}{2}(\gamma_1+\gamma_2)}\log n) \to 0 \text{ as } n \to \infty.$$

This is due to the fact that $\gamma_1 + \gamma_2 \geq 1$. The desired result is now immediate from Lemma 3.
For the interpolatory projection, we need to assume that $\alpha \geq 1$ and $\gamma \geq 0$. In other words, $\gamma_1 \geq 1$. Let $x \in \mathcal{C}[-1,1]$. Since $x - \pi_n^C x \in \mathcal{C}[-1,1]$, it follows that $T(I - \pi_n^C)x \in \mathcal{C}^{\gamma_1}[-1,1]$. Hence, the bounds (2.9) and (2.11) imply that

$$\begin{aligned} \|(T - T_n^M)x\|_\infty &\leq C_1 n^{\frac{1}{2}-\gamma_1}\|(T(I - \pi_n^C)x)^{(\gamma_1)}\|_\infty \\ &\leq C_1 A_1 n^{\frac{1}{2}-\gamma_1}\|(I - \pi_n^C)x\|_{L^2}, \end{aligned} \tag{3.12}$$

where

$$A_1 = \max_{s\in[-1,1]} \left[\int_{-1}^{1} \left|\frac{\partial^{\gamma_1}\kappa}{\partial s^{\gamma_1}}(s,t)\right|^2 dt\right]^{\frac{1}{2}}.$$

Consequently,

$$\|T - T_n^M\|_\infty \leq \sqrt{2}C_1(1 + p)A_1 n^{\frac{1}{2}-\gamma_1} \to 0 \text{ as } n \to \infty.$$

The proof is complete. □

Remark 1. In the case of the interpolatory projection, it can be easily checked from (2.6) and (2.7) that, T_n^M is $\nu-$convergent to T in L^2-norm for all $\gamma \geq -1$. This means that

$$\sup_{n\geq N} \|(I - T_n^M)^{-1}\|_2 \leq c < \infty.$$

Now, we have from (2.3) and (3.10),

$$\|T(I - T_n^M)^{-1}\|_\infty \leq C_3\|T\|_\infty < \infty.$$

Then, the following two results are quoted from [12] and [15] respectively.

$$\|u - u_n^M\|_\infty \le \|(I - T_n^M)^{-1}\|_\infty \|(I - \pi_n)T(I - \pi_n)u\|_\infty, \tag{3.13}$$

$$\|u - \widetilde{u}_n^M\|_\infty \le \left\{1 + \sqrt{2}p\|T(I - T_n^M)^{-1}\|_\infty\right\} \|T(I - \pi_n)T(I - \pi_n)u\|_\infty. \tag{3.14}$$

We immediately, obtain

$$\|u - u_n^M\|_\infty \le C_3(1 + \|\pi_n\|_\infty)\|T(I - \pi_n)u\|_\infty, \tag{3.15}$$

$$\|u - u_n^M\|_\infty \le C_3\|(I - \pi_n)T\|_\infty \|(I - \pi_n)u\|_\infty, \tag{3.16}$$

$$\|u - \widetilde{u}_n^M\|_\infty \le C_2\|T(I - \pi_n)T\|_\infty \|(I - \pi_n)u\|_\infty. \tag{3.17}$$

The above bounds will allow us to establish several rates of convergence for u_n^M and $\widetilde{u}_n^M$ and to deduce then the optimal ones.

3.1 Galerkin and modified Galerkin methods

The following result is crucial.

Theorem 2. *Let T be an integral operator with kernel $\kappa \in \mathcal{C}(\alpha, \gamma)$. For $\mu = 0, \ldots, \alpha$, if $x \in \mathcal{C}^\mu[-1, 1]$, then for n large enough*

$$\|T(I - \pi_n^G)x\|_\infty \le Cn^{-\mu - \frac{1}{2}(\gamma_1 + \gamma_2)}\|x^{(\mu)}\|_{L^2}. \tag{3.18}$$

Proof. For each $s \in [-1, 1]$, by (3.11) and the Cauchy-Schwarz inequality, we obtain

$$\left|[T(I - \pi_n^G)x](s)\right| \le \|(I - \pi_n^G)\kappa_s\|_{L^2}\|(I - \pi_n^G)x\|_{L^2}.$$

Taking supremum over $s \in [-1, 1]$, and using (2.8), (3.8) show the desired result. □

In the rest of this section, we assume for the projection methods that $\kappa \in \mathcal{C}(\alpha, \gamma)$, while for modified projection method it is assumed that $\kappa \in \mathcal{C}(2\alpha, \gamma)$. It is to be noted that the analysis of the error for u_n^M and $\widetilde{u}_n^M$ is adapted from Kulkarni [13].

Theorem 3. *Assume that $f \in \mathcal{C}^\alpha[-1, 1]$. Let u_n^G and $\widetilde{u}_n^G$ be the Galerkin and the iterated Galerkin solutions respectively. Then, for n large enough,*

$$\|u - u_n^G\|_\infty = \mathcal{O}(n^{-\alpha + \frac{3}{4}}), \tag{3.19}$$

$$\|u - \widetilde{u}_n^G\|_\infty = \mathcal{O}(n^{-\alpha - \frac{1}{2}(\gamma_1 + \gamma_2)}), \tag{3.20}$$

$$\|u - u_n^M\|_\infty = \mathcal{O}(n^{-\alpha - \frac{1}{2}(\gamma_1 + \gamma_2)} \log n), \tag{3.21}$$

$$\|u - \widetilde{u}_n^M\|_\infty = \mathcal{O}(n^{-\alpha - \gamma_1 - \gamma_2}). \tag{3.22}$$

Proof. The estimate (3.19) is obtained by combining (2.9) and (3.2), while (3.20) is a consequence of (3.3) and (3.18). The operator T is a continuous operator from $\mathcal{C}^\alpha[-1,1]$ into $\mathcal{C}^\alpha[-1,1]$. Then, as $u \in \mathcal{C}^\alpha[-1,1]$ we have $T(I-\pi_n^G)u \in \mathcal{C}^\alpha[-1,1] \subset \mathcal{C}^{\gamma_1}[-1,1]$. It follows from (2.9) that

$$\|(I-\pi_n^G)T(I-\pi_n^G)u\|_\infty \leq C_1 n^{-\gamma_1+\frac{3}{4}}\|(T(I-\pi_n^G)u)^{(\gamma_1)}\|_\infty. \tag{3.23}$$

Noting that

$$(T(I-\pi_n^G)u)^{(\gamma_1)}(s) = \int_{-1}^{1} q(s,t)(I-\pi_n^G)u(t)dt,$$

where the kernel q is of class $\mathcal{C}(\alpha, \gamma-\gamma_1)$, the estimate (3.18) gives,

$$\|(T(I-\pi_n^G)u)^{(\gamma_1)}\|_\infty \leq C n^{-\alpha-\frac{1}{2}(\min\{\alpha,\gamma-\gamma_1+1\}+\min\{\alpha,\gamma-\gamma_1+2\})}\|x^{(\alpha)}\|_{L^2}. \tag{3.24}$$

By combining (3.23) and (3.24) with (3.13), we obtain

$$\|u-u_n^M\|_\infty = \mathcal{O}(n^{\frac{3}{4}-\alpha-\frac{1}{2}(\gamma_1+\gamma_2)}).$$

We observe that the required order of convergence in (3.21) is higher and is derived simply from (2.10), (3.18) and (3.15).

By (3.18) and (3.24), one gets

$$\begin{aligned}\|T(I-\pi_n^G)T(I-\pi_n^G)u\|_\infty &\leq C n^{-\gamma_1-\frac{1}{2}(\gamma_1+\gamma_2)}\|(T(I-\pi_n^G)u)^{(\gamma_1)}\|_{L_2}\\ &\leq C n^{-\alpha-\gamma_1-\gamma_2}\|x^{(\mu)}\|_{L^2}\end{aligned}$$

and the estimate (3.14) shows (3.22). This order is better than the one that can be achieved using the error bound (3.17). Indeed, by (3.11), (2.13) and (3.8) we have for each $x \in L^2[-1,1]$ and each $s \in [-1,1]$

$$\begin{aligned}|T(I-\pi_n^G)Tx(s)| &\leq \|(I-\pi_n^G)\kappa_s\|_{L^2}\|(I-\pi_n^G)Tx\|_{L^2}\\ &\leq C_1 n^{-\frac{1}{2}(\gamma_1+3\gamma_2)}\|(Tx)^{(\gamma_2)}\|_{L^2} \leq \sqrt{2}C_1 K n^{-\frac{1}{2}(\gamma_1+3\gamma_2)}\|x\|_\infty.\end{aligned}$$

As a consequence,

$$\|T(I-\pi_n^G)T\|_\infty = \mathcal{O}(n^{-\frac{1}{2}(\gamma_1+3\gamma_2)}).$$

This together with (2.9) show that

$$\|u-\widetilde{u}_n^M\|_\infty = \mathcal{O}(n^{-\alpha-\frac{1}{2}(\gamma_1+3\gamma_2)+\frac{3}{4}}).$$

Taking into account that $\gamma_1+1 \geq \gamma_2$, the remark $\gamma_1+\gamma_2 \geq \frac{1}{2}(\gamma_1+3\gamma_2)-\frac{3}{4}$, ends the proof. $\square$

Remark 2. From Theorem 3, we observe that while u_n^M and $\widetilde{u}_n^G$ have almost the same rate of convergence, the solution $\widetilde{u}_n^M$ is more accurate than both of them. We see also that $\widetilde{u}_n^G$, is more accurate than u_n^G. If $\alpha > \gamma+1$, we obtain the orders

$$\begin{aligned}&\|u-u_n^G\|_\infty = \mathcal{O}(n^{-\alpha+\frac{3}{4}}), \quad \|u-\widetilde{u}_n^G\|_\infty = \mathcal{O}(n^{-\alpha-\gamma-\frac{3}{2}}),\\ &\|u-u_n^M\|_\infty = \mathcal{O}(n^{-\alpha-\gamma-\frac{3}{2}}\log n), \quad \|u-\widetilde{u}_n^M\|_\infty = \mathcal{O}(n^{-\alpha-2\gamma-3}).\end{aligned}$$

3.2 Collocation and modified collocation methods

Theorem 4. *Let T be an integral operator with kernel $\kappa \in \mathcal{C}(\alpha,\gamma)$. For $\mu = 0,\dots,\alpha$, if $x \in \mathcal{C}^{\mu}[-1,1]$, then*

$$\|M(I-\pi_n^C)v\|_\infty \le Cn^{-\mu}\|v^{(\mu)}\|_2. \tag{3.25}$$

Proof. Since

$$[T(I-\pi_n^C)v](s) = \int_{-1}^{1} m(s,t)(v-\pi_n^C v)(t)dt, \quad s \in [-1,1].$$

By taking the supremum over $[-1,1]$, we get

$$\|M(I-\pi_n^C)v\|_\infty \le \max_{s\in[-1,1]} \|m_s\|_2\|v-\pi_n^C v\|_2 \le \sqrt{2}C_1M_0n^{-\mu}\|v^{(\mu)}\|_2.$$

Whence, (3.25) is proved. □

Theorem 5. *Let $\gamma \ge 0$ and assume that $f \in \mathcal{C}^{\alpha}[-1,1]$ with $\alpha \ge 1$. Let u_n^C and $\widetilde{u}_n^C$ be the collocation and the iterated collocation solutions respectively. Then, for n large enough,*

$$\|u-u_n^C\|_\infty = \mathcal{O}(n^{-\alpha+\frac{1}{2}}), \tag{3.26}$$
$$\|u-\widetilde{u}_n^C\|_\infty = \mathcal{O}(n^{-\alpha}), \tag{3.27}$$
$$\|u-u_n^M\|_\infty = \mathcal{O}(n^{-\alpha-\gamma_2+1}), \tag{3.28}$$
$$\|u-\widetilde{u}_n^M\|_\infty = \mathcal{O}(n^{-\alpha-\gamma_2+\frac{1}{2}}). \tag{3.29}$$

Proof. The estimate (3.26) follows from (2.9) and (3.2), while (3.27) follows from (3.25) and (3.3). We notice that if we use (3.15) and (3.25), we get only the order $\mathcal{O}(n^{-\alpha+\frac{1}{2}})$ for u_n^M. This order can be improved and reach a $\mathcal{O}(n^{-\alpha-\gamma_2+1})$, if the bound (3.16) is used. Indeed, by (2.9) and by (2.3)

$$\|(I-\pi_n^C)Tx\|_\infty \le C_1n^{-\gamma_2+\frac{1}{2}}\|(Tx)^{(\gamma_2)}\|_\infty \le Cn^{-\gamma_2+\frac{1}{2}}\|x\|_\infty,$$

which leads to

$$\|(I-\pi_n^C)T\|_\infty = \mathcal{O}(n^{-\gamma_2+\frac{1}{2}}).$$

The required result follows from (2.9) and the above estimate. Now, the optimal order is recovered by using (3.12) and (2.8).

Note that by (2.8) and (2.13),

$$\|(I-\pi_n^C)Tx\|_{L^2} \le C_1n^{-\gamma_2}\|(Tx)^{(\gamma_2)}\|_{L^2} \le C_1Kn^{-\gamma_1-1}\|x\|_{L^2}.$$

Hence $\|(I-\pi_n^C)T\|_{L^2} = \mathcal{O}(n^{-\gamma_2})$ and therefore

$$\|T(I-\pi_n^C)T\|_\infty = \mathcal{O}(n^{-\gamma_2}). \tag{3.30}$$

Here we have used

$$|T(I-\pi_n^C)Tx(s)| \le A_1\|(I-\pi_n^C)Tx\|_{L^2},$$

which is equivalent to

$$\|T(I-\pi_n^C)T\|_\infty \leq A_1\|(I-\pi_n^C)T\|_{L^2}.$$

Combining (3.30) and (2.9) with (3.17) ends the proof of (3.29). It is to be noted that when we combine (3.25) with (3.14), we obtain only the order $\mathcal{O}(n^{-\gamma_1-\alpha})$ which is slower than (3.29). □

Remark 3. In general, the iterated modified solution $\widetilde{u}_n^M$ is the fastest solution and the collocation solution is the slowest. Moreover, the modified collocation solution u_n^M converge faster than the iterated collocation solution $\widetilde{u}_n^C$. For $\gamma = 0$, we obtain

$$\|u-u_n^C\|_\infty = \mathcal{O}(n^{-\alpha+\frac{1}{2}}), \quad \|u-\widetilde{u}_n^C\|_\infty = \mathcal{O}(n^{-\alpha}),$$
$$\|u-u_n^M\|_\infty = \mathcal{O}(n^{-\alpha-1}), \quad \|u-\widetilde{u}_n^M\|_\infty = \mathcal{O}(n^{-\alpha-\frac{3}{2}}).$$

Noting that $\pi_n^C u_n = \pi_n^C \widetilde{u}_n$ and $\pi_n^C u_n^M = \pi_n^C \widetilde{u}_n^M$, which implies that at the collocation nodes, the collocation and the modified collocation methods converge with the same speed as their iterated versions.

Remark 4. It was also possible to use the collocation at the Gauss-Radau, the Gauss-Lobatto points or even at Tchebychev points. In this case, we expect the same convergence orders obtained previously.

4 Numerical results

In this section, a numerical example is given to illustrate the theory established in the previous sections. We consider the following integral equation quoted from [13]

$$u(s) - \int_0^1 \kappa(s,t)u(t)dt = f(s), \quad s \in [-1,1],$$

where

$$\kappa(s,t) = \begin{cases} s(1-t), & s \leq t, \\ t(1-s), & t \leq s, \end{cases}$$

and $f(s)$ is selected so that $u(s) = s^{\frac{9}{2}}$. Thus, $u \in \mathcal{C}^4[0,1]$ and $u \notin \mathcal{C}^5[0,1]$. In this example we have

$$\alpha = 4, \quad \gamma = 0, \quad \gamma_1 = 1, \quad \gamma_2 = 2.$$

Let $\mathbb{X}_n$ be the space of polynomials of degree $\leq n$. The computations are done for $2 \leq n \leq 8$. Note that, all required integrals were calculated by a Gauss-quadrature rule. We present the errors obtained by the modified projection method and its iterated version and we compare them with those obtained by the projection and the iterated projection methods. The results are given in Tables 1–2.

Even though the iterated projection and the modified projection methods have the same rate, we observe, in conformity with the theory, that the iterated

Table 1. Orthogonal projection.

n	$\|u-u_n^G\|_\infty$	$\|u-\widetilde{u}_n^G\|_\infty$	$\|u-u_n^M\|_\infty$	$\|u-\widetilde{u}_n^M\|_\infty$
2	1.45877×10^{-1}	1.13050×10^{-3}	9.51061×10^{-4}	1.32928×10^{-5}
3	2.58135×10^{-2}	1.31364×10^{-4}	1.00386×10^{-4}	9.74411×10^{-7}
4	1.51176×10^{-4}	4.10769×10^{-6}	3.72095×10^{-6}	2.03446×10^{-8}
5	8.64590×10^{-5}	1.42896×10^{-7}	1.40984×10^{-7}	5.21080×10^{-10}
6	1.31632×10^{-5}	1.33924×10^{-8}	1.50929×10^{-8}	3.64242×10^{-11}
7	3.00966×10^{-6}	2.03645×10^{-9}	2.53317×10^{-9}	4.41795×10^{-12}
8	8.77996×10^{-7}	4.14569×10^{-10}	5.60523×10^{-10}	7.11813×10^{-13}

Table 2. Interpolatory projection.

n	$\|u-u_n^C\|_\infty$	$\|u-\widetilde{u}_n^C\|_\infty$	$\|u-u_n^M\|_\infty$	$\|u-\widetilde{u}_n^M\|_\infty$
2	1.65449×10^{-1}	1.52518×10^{-3}	8.68407×10^{-4}	1.95506×10^{-5}
3	2.69583×10^{-2}	1.31524×10^{-4}	1.00361×10^{-4}	2.30886×10^{-6}
4	1.58336×10^{-3}	4.13678×10^{-6}	3.40545×10^{-6}	3.47622×10^{-8}
5	9.73069×10^{-5}	1.35795×10^{-7}	1.16787×10^{-7}	8.42624×10^{-10}
6	1.56602×10^{-5}	1.30225×10^{-8}	1.12519×10^{-8}	5.52156×10^{-11}
7	3.74297×10^{-6}	1.94947×10^{-9}	1.69288×10^{-9}	6.24665×10^{-12}
8	1.13231×10^{-6}	3.85032×10^{-10}	3.34933×10^{-10}	9.88022×10^{-13}

modified projection approximation converges the fastest than both of them. We believe that sharper estimates than those stated previously could have been provided, especially in the case of the interpolatory projection.

Let

$$\tau_j = \cos\left[\frac{2j-1}{2n}\pi\right], \quad j = 1, \ldots, n.$$

These points are known as the classical Chebyshev points, and are the zeros of the n^{th} degree Tchebychev polynomial of the first kind T_n, defined by

$$T_n(\cos n\theta) = \cos n\theta.$$

By using these points, we obtain the results given in Table 3.

Table 3. Collocation at Tchebychev points.

n	$\|u-u_n^C\|_\infty$	$\|u-\widetilde{u}_n^C\|_\infty$	$\|u-u_n^M\|_\infty$	$\|u-\widetilde{u}_n^M\|_\infty$
2	1.03409×10^{-1}	2.55726×10^{-3}	1.53757×10^{-3}	3.90372×10^{-5}
3	1.47021×10^{-2}	3.56504×10^{-4}	1.72021×10^{-4}	5.72928×10^{-6}
4	7.82493×10^{-3}	8.33308×10^{-6}	5.46370×10^{-6}	7.03910×10^{-8}
5	4.47585×10^{-5}	2.49501×10^{-7}	1.88432×10^{-7}	1.89603×10^{-9}
6	6.81413×10^{-6}	2.11725×10^{-8}	1.79098×10^{-8}	1.09988×10^{-10}
7	1.55786×10^{-6}	3.33072×10^{-9}	2.63310×10^{-9}	1.15512×10^{-11}
8	4.54432×10^{-7}	6.64968×10^{-10}	5.37712×10^{-10}	1.79301×10^{-12}

It can be seen that the collocation at the Gauss points is slightly better than the corresponding one at the Tchebychev points.

5 Conclusions

The above tables illustrate that high accuracy is obtained by the proposed methods even when the polynomials are of low degree and the exact solution with limited smoothness. It should be mentioned that to obtain an accuracy of comparable order by piecewise polynomials a very much larger linear systems are needed to be solved. For example, to obtain the error of order 10^{-13} in the iterated modified projection method with $n = 8$, a system of size 9×9 is needed to be solved, whereas in the piecewise polynomial basis (see [13]) we need to solve a system of size 256×256. We feel that the methods proposed in this paper can be extended naturally to nonlinear Urysohn integral equations and also to eigenvalue problems. We expect that for the special case of Hammerstein equations the Legendre superconvergent projection-type methods studied in [2] will give the same performance as the modified projection methods.

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