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EARTH SECTION PATHS. SOLUTION TO THE INVERSE AND DIRECT PROBLEMS, AND WAYPOINTS WITHOUT ITERATIONS

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Abstract. The use of central elliptical sections in the calculation of air and sea routes and normal sections in Geodesy is common. Elliptic sections do not represent the shortest path between two points, although are often used in navigation to replace the geodesic lines. All developments include some kind of iteration to solve one of the problems, direct or inverse. When using vector algebra methods and perturbed series, the problems can be solved using an equidistant circle that represents the path of the elliptical section. This is possible because the flattening of the elliptical section is less than or equal to the Earth's flattening, which implies that the series calculated for the terrestrial ellipsoid, used in the section, always converge. Three direct methods are described in order to calculate: the distance, the azimuth, the coordinates of a point and intermediate positions of an elliptical section. Those algorithms provide solutions to the inverse and direct algorithms with a consistency of the order of truncation error of double-type numbers.

Keywords: elliptical sections, routes, waypoints, great ellipse, direct formula, rectified angle, elliptic arc.

Introduction

An elliptical section is the curve that results from the intersection of a plane with the ellipsoid of revolution. For an Earth section path, that plane contains three points: two of them on the surface of the ellipsoid and the third is determined by the type of the section. Three cases are studied, illustrated in Figures 1 and 2: the normal section, the mean normal section and the central section. The normal section is determined by the plane that contains the normal line at one of the points of interest (P_1 to P_0^1 in Figure 1), in addition to the two points on the surface (P_1 and P_2). Since normal lines at two points are not coplanar, there are two normal sections: one that includes the

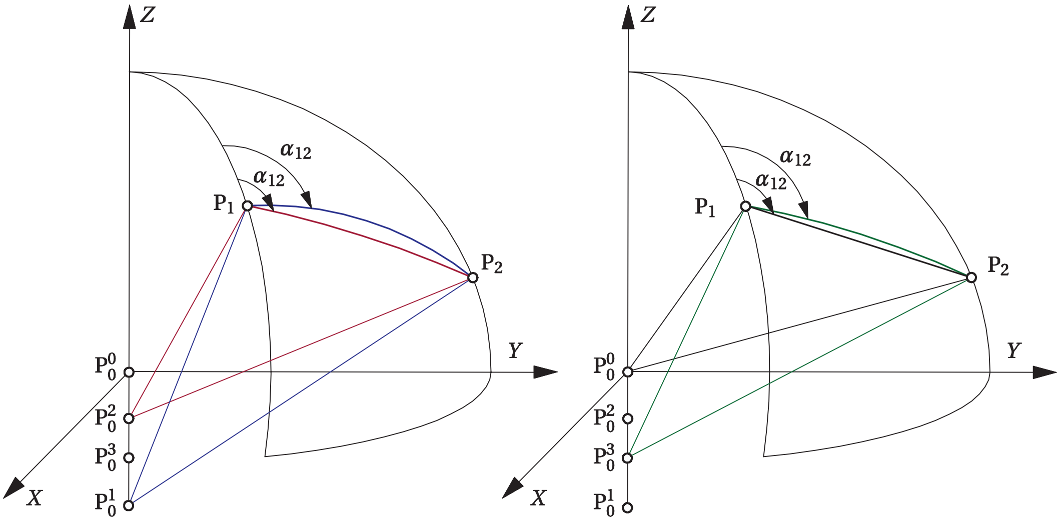

Figure 1. Schema of elliptical sections; on the left: the first (in blue) and second section (in red); on the right: the mean (in green) and central section (in black)

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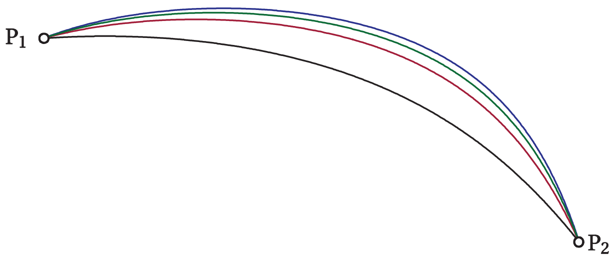

Figure 2. Representation in a *plate carrée* projection of elliptical sections with coordinate points $\phi_1 = 60°$, $\lambda_1 = 0°$ and $\phi_2 = 30°$, $\lambda_2 = 80°$. From bottom to top: great ellipse, second normal section, mean normal section and first normal section. The ellipsoid flattening was exaggerated at $f = 1/2$

normal line of P_1 and another that includes the normal line of P_2, (P_2 to P_0^2 in Figure 1) with the exception of cases where the points are on the same meridian or parallel. The mean normal section is between the previous two, the third point of the plane is located at a midpoint of the intersections of the two normal lines on the Z-axis (P_0^3 in Figure 1). This definition is different from that given by Gilbertson(2012); later, the topic will be addressed. Finally, in the central section or great ellipse, the plane contains the center of the ellipsoid, P_1 and P_2, being a unique case since the straight lines from the origin to the points are coplanar. In the event the points have latitudes of different signs, the central elliptical section is between the first and second normal sections.

Unlike geodesic lines, the elliptical sections are plane curves and can be easily solved. The geodesic line is the shortest path on the ellipsoid between two points. The most convenient approach for the solution of the direct and inverse problems for geodesics is given by Karney (2013). When the two points are on the equator or on a meridian, all the elliptical sections match with the path of the geodesic line; these are the only cases where there is a flat curve.

The simplicity of the geometric problem is the main advantage of solving paths that connect two points on the ellipsoid with elliptical sections instead of geodesic lines. The normal section is usually used in Geodesy because it's easier to reduce a raw EDM (Electronic Distance Measurement) distance to the normal section than to the geodesic line. These measurements must be corrected due to the vertical deflection and ellipsoidal height. The great ellipse is studied in navigation by analogy of the great circle, or orthodromic line in the terrestrial sphere; however, the path of the great ellipse differs more from the geodesic line than do normal sections, specially if the points are on the same hemisphere away from the equator.

The problem of elliptical terrestrial sections was extensively addressed both by Geodesy and Maritime Navigation. Literature reviews and different solutions to the problems can be seen in Deakin (2009), Sjöberg (2012), Gilbertson (2012) and Tseng et al. (2013), among others. In all cases, they start with the intersection of a plane with the ellipsoid; the parameters of the resulting ellipse are found and the inverse and direct problems are solved. The inverse problem is to calculate the distance and azimuth of the elliptical section, given the coordinates of the points, and the direct problem to find the coordinates of a second point, given the coordinates of the first and the distance and azimuth of the section. Iterative processes are used in both the case of the direct problem and the search for equidistant points of the section path.

All the intersections of a plane with an ellipsoid of revolution are circles or ellipses (Deakin, 2009, pp. 3–5). The flattening of the normal and central sections is equal to or less than the flattening of the ellipsoid. Therefore, it is possible to apply the same perturbed series used in the terrestrial ellipsoid. In particular, relationships between geodetic, parametric and equidistant latitudes can be obtained without resorting to the numerical resolution of integrals or iterative processes. The length of an elliptic arc is simply the product between its equidistant radius and the difference among the rectified latitudes. This technique is the one that will be used to solve the problems: first, calculate the elliptical section and rotate it into a plane coordinate system, then transform the ellipse to an equidistant circumference. Any direct or inverse problem of distance can be solved in this circumference, and then the coordinates are transformed again to the original system. The use of spherical trigonometry formulas is completely unnecessary; only algebra and plane trigonometry are necessary.

This work does not try to demonstrate that the elliptical sections can replace the geodesic line, but to obtain a "closed" solution to the elliptical sections, mainly in obtaining waypoints.The following paragraphs explain the method used and describe algorithms and formulas to solve the inverse problem, the direct problem and waypoints of the route. In addition, numerical examples are provided comparing the results obtained for the different elliptical sections.

1. Description of the method

In all cases, λ_1 is taken as central meridian. In the direct method and the construction of waypoints, λ_2 is corrected by adding the original value of λ_1.

In the first three subsections(1.1 to 1.3) well-known formulas are described. In subsection 1.4, Gilbertson formulas (2012) are used; from subsection 1.5, the formulas of the proposed method are developed.

This method fails when P_1 and P_2 are simultaneously on the equator; in this case, the problem is solved on a circle of radius a using λ_1 and λ_2 as coordinates, both α_{12} and α_2 are equal to $\pm\pi/2$.

1.1. Cartesian coordinates of the points

In order to find the elliptical section, we must first calculate the ECEF coordinates of three points that belong to the plane.

Two points: P_1 and P_2, are the sites of interest on the surface of the ellipsoid,

$$\mathrm{P}_1 = \left[X_1, Y_1, Z_1\right]^{\mathrm{T}}; \quad (1)$$

$$\mathrm{P}_2 = \left[X_2, Y_2, Z_2\right]^{\mathrm{T}}, \quad (2)$$

where

$$X = R_N \cos\phi \cos\lambda;$$
$$Y = R_N \cos\phi \sin\lambda; \quad (3)$$
$$Z = R_N \left(1 - e^2\right) \sin\phi,$$

ϕ is geodetic latitude, λ is geodetic longitude, e is first eccentricity of the ellipsoid and R_N is the radius of curvature in the prime vertical.

The ellipsoid of revolution is determined by the semi-major axis a and the first flattening f. For example, in ellipsoid GRS80, $a = 6\,378\,137$ m and $1/f = 298.2572221008827$. The first eccentricity e is defined as

$$e^2 = f\left(2 - f\right),$$

and the radius of curvature in the prime vertical is calculated as

$$R_N = \frac{a}{\sqrt{1 - e^2 \sin^2\phi}}.$$

The position of the third point is determined by the intersection of the straight lines, passing through P_1 and P_2, with the Z-axis,

$$\mathrm{P}_0 = \left[0, 0, Z_0\right]^{\mathrm{T}}, \quad (4)$$

where Z_0 depends on the elliptical section we are considering:

1. First normal section,

$$Z_0 = -e^2 \sin\phi_1 \, R_N\big|_1.$$

2. Second normal section,

$$Z_0 = -e^2 \sin\phi_2 \, R_N\big|_2.$$

3. Mean normal section,

$$Z_0 = \frac{-e^2}{2}\left(\sin\phi_1 \, R_N\big|_1 + \sin\phi_2 \, R_N\big|_2\right).$$

Gilbertson (2012) defines the construction of the mean normal section through the average of unit vectors at P_1 and P_2, instead of considering an average position of the intersections of the normals with the Z-axis. However, it is preferable to define the elliptic sections consistently taking into account three points, one of which will always be on the axis of rotation Z. In practice, the different ways of defining the meannormal section have very small inconsistencies,as discussed below.

4. Great ellipse,

$$Z_0 = 0.$$

1.2. Plane of the elliptical section

Points P_1, P_2 and P_0 determine two unit vectors that belong to the plane of the elliptical section,

$$\hat{\mathrm{u}} = \frac{\mathrm{P}_1 - \mathrm{P}_0}{\left\|\mathrm{P}_1 - \mathrm{P}_0\right\|}, \quad (5)$$

$$\hat{\mathrm{v}} = \frac{\mathrm{P}_2 - \mathrm{P}_0}{\left\|\mathrm{P}_2 - \mathrm{P}_0\right\|}. \quad (6)$$

The unit vector perpendicular to the plane is calculated as

$$\hat{\mathrm{n}} = \frac{\hat{\mathrm{u}} \times \hat{\mathrm{v}}}{\left\|\hat{\mathrm{u}} \times \hat{\mathrm{v}}\right\|}. \quad (7)$$

The equation of the plane is determined by a point thereof and a perpendicular vector. Taking as a point belonging to plane P_0, according to the properties of the vector scalar product, and being $\hat{\mathrm{n}} = \left[n_1, n_2, n_3\right]^{\mathrm{T}}$ and $\mathbf{X} = \left[X, Y, Z\right]^{\mathrm{T}}$ the coordinates of any other point belonging to the plane,

$$\left(\mathrm{P}_0 - \mathbf{X}\right) \cdot \hat{\mathrm{n}} = 0.$$

$$\mathbf{X} \cdot \hat{\mathrm{n}} = \mathrm{P}_0 \cdot \hat{\mathrm{n}},$$

expanding the formula,

$$X n_1 + Y n_2 + Z n_3 = d, \quad (8)$$

where

$$d = X_0 n_1 + Y_0 n_2 + Z_0 n_3. \quad (9)$$

1.3. Azimuth of a point on the ellipse

At any point on the ellipsoid,

$$\hat{\mathrm{E}} = \left[-\sin\lambda, \cos\lambda, 0\right]^{\mathrm{T}},$$
$$\hat{\mathrm{N}} = \left[-\sin\phi\cos\lambda, -\sin\phi\sin\lambda, \cos\phi\right]^{\mathrm{T}}, \quad (10)$$
$$\hat{\mathrm{U}} = \left[\cos\phi\cos\lambda, \cos\phi\sin\lambda, \sin\phi\right]^{\mathrm{T}},$$

are unit vectors that point East, North and Up, respectively. To find the azimuth, we require a vector tangent to the curve of the elliptical section. The tangent vector at any point can be calculated as

$$\vec{\mathrm{t}} = \hat{\mathrm{n}} \times \hat{\mathrm{U}}, \quad (11)$$

where $\hat{\mathrm{n}}$ is the vector perpendicular to plane, formula (7). The East component of the tangent vector is $\vec{\mathrm{t}} \cdot \hat{\mathrm{E}}$ and the North component $\vec{\mathrm{t}} \cdot \hat{\mathrm{N}}$, then the azimuth is

$$\alpha = \arg\left(\vec{\mathrm{t}} \cdot \hat{\mathrm{N}}, \vec{\mathrm{t}} \cdot \hat{\mathrm{E}}\right). \quad (12)$$

The function $\arg\left(x, y\right) = \arg\left(x + \mathrm{i} \cdot y\right)$ is equivalent to the Fortran function $\mathrm{atan2}\left(y, x\right)$.

The vector tangent to the elliptical section given an azimuth α_1 is

$$\hat{\mathrm{t}} = \hat{\mathrm{N}} \cos\alpha_1 + \hat{\mathrm{E}} \sin\alpha_1. \quad (13)$$

1.4. Parameters of the elliptical section

The intersection of the plane with the ellipsoid always results in an ellipse that, in turn, represents the desired path. The values of the semi-axes and the coordinates of the center of this ellipse are required.

Gilbertson (2012, p. 2) proposes the following solution for the parameters of the ellipse:

Given

$$p=\sqrt{n_1^2+n_2^2}, \tag{14}$$

$$C=b^2n_3^2+a^2p^2,\quad E=dn_3p\left(a^2-b^2\right),$$

where n_1, n_2 and n_3 are the components of vector $\hat{\mathrm{n}}$. The base vectors of the rotated coordinate system are as follows:

$$\hat{\mathrm{i}}=\begin{bmatrix} n_2/p \\ -n_1/p \\ 0 \end{bmatrix},\quad \hat{\mathrm{j}}=\begin{bmatrix} n_1n_3/p \\ n_2n_3/p \\ -p \end{bmatrix},\quad \hat{\mathrm{k}}=\hat{\mathrm{n}}.$$

With this data, the coordinates of the center of the ellipse are

$$\mathbf{x}_0=\frac{E}{C}\hat{\mathrm{j}}+d\hat{\mathrm{k}}, \tag{15}$$

and the semi-axes and derived parameters from the flattening are calculated as

$$\dot{a}=a\sqrt{1-\frac{d^2}{C}},\quad \dot{b}=\dot{a}\frac{b}{\sqrt{C}}, \tag{16}$$

$$\dot{f}=\frac{\dot{a}-\dot{b}}{\dot{a}},\quad \dot{e}^2=\dot{f}\left(2-\dot{f}\right),\quad \dot{n}=\frac{\dot{f}}{2-\dot{f}}, \tag{17}$$

where $\dot{n}$ is the third flattening of the elliptical section.

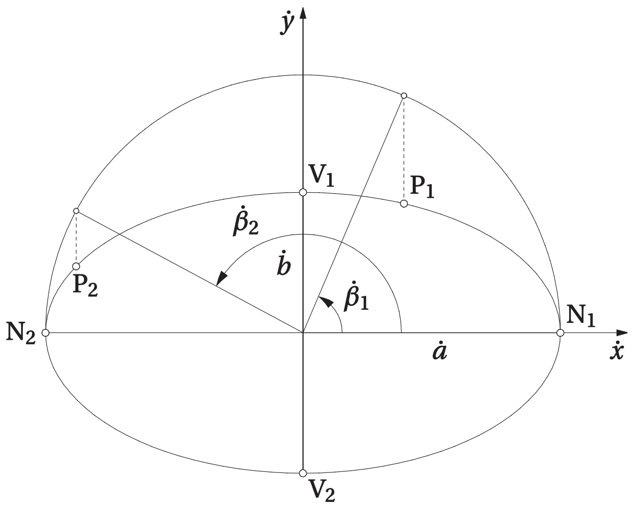

Figure 3. Elliptical section in a plane coordinate system $\dot{x}\dot{y}$. Vertices, nodes and the parametric angles of points P_1 and P_2 are indicated by V_1, V_2, N_1, N_2 and β_1 and β_2, respectively

1.5. Orientation of the ellipse in a plane coordinate system

ECEF coordinates are moved and rotated, so that semi-axis b matches a local $\dot{y}$-axis, and semi-axis $\dot{a}$ matches an $\dot{x}$-axis. The rotation matrices are obtained with the parameters of the perpendicular vector to the plane in such way that it coincides with the $\dot{x}\dot{y}$-plane.

First, the Z-axis is rotated according to the orientation of $\hat{\mathrm{n}}$, plus an additional $\pi/2$ rotation so that the X-axis points towards the local "North". Being $\sin\varepsilon_Z=n_2/p$ and $\cos\varepsilon_Z=n_1/p$,

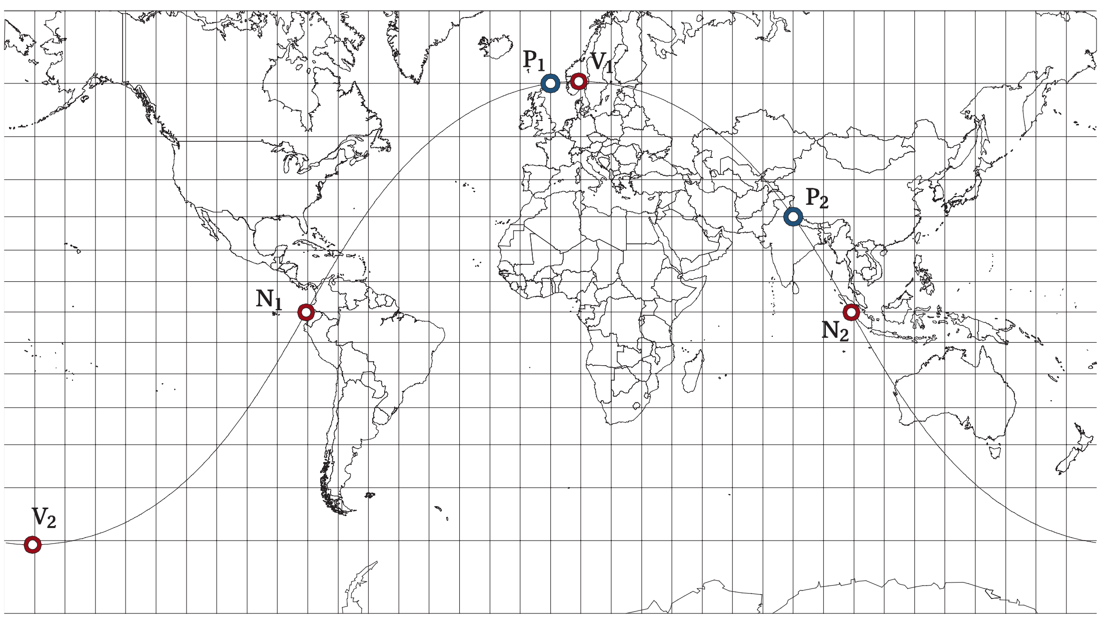

Figure 4. First normal section in a Mercator chart. The same points of the example in Figure 2 are used. The nodes and vertices of the path are indicated. The flattening of the ellipsoid is the original of GRS80. The path is closed when one or more full turns of the ellipsoid are made

$$\mathbf{R}_Z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_1/p & n_2/p & 0 \\ -n_2/p & n_1 p & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -n_2/p & n_1/p & 0 \\ -n_1/p & -n_2/p & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where p is from formula (14).

Then the X -axis is rotated to null the Z coordinate. Being $\sin\varepsilon_X = n_3$ and $\cos\varepsilon_X = p$,

$$\mathbf{R}_X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & n_3 & p \\ 0 & -p & n_3 \end{bmatrix}.$$

Finally, the conversion of ECEF coordinates to a plane system containing the ellipse is:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \mathbf{R}_X \mathbf{R}_Z \begin{bmatrix} X - x_0 \\ Y - y_0 \\ Z - z_0 \end{bmatrix}. \quad (18)$$

The $\dot{z}$ coordinate should always result in a null value.

This system is illustrated in Figure 3; the parametric angles of the points on the ellipse, $\dot{\beta}_1$ and $\dot{\beta}_2$, arise from the equations

$$\dot{x} = \dot{a}\cos\dot{\beta},$$
$$\dot{y} = \dot{b}\sin\dot{\beta},$$

thus

$$\dot{\beta} = \arg\left(\frac{\dot{x}}{\dot{a}}, \frac{\dot{y}}{\dot{b}}\right), \quad (19)$$

where $-\pi < \dot{\beta} \le \pi$. As an example, the projection of the first normal section, along with vertices and nodes, is illustrated in Figure 4.

1.6. Construction of the equidistant circumference

The equidistant circumference has an equidistant radius $\dot{R}_\mu$ and rectified angles $\dot{\mu}$ transformed from parametric angles $\dot{\beta}$. This can be achieved by analogy to the parametric and rectified latitudes in the ellipsoid of revolution, using the same formulas.

$$\dot{\mu} = \dot{\beta} + \sum_{j=1}^{j=6} D_j \sin\left(2j\dot{\beta}\right), \quad (20)$$

where $\dot{R}_\mu$ and coefficients D_j are calculated according to Table 1, by replacing a with $\dot{a}$ and n with $\dot{n}$. To differentiate this coordinate system with that of the ellipse, we rename it to ξ, η .

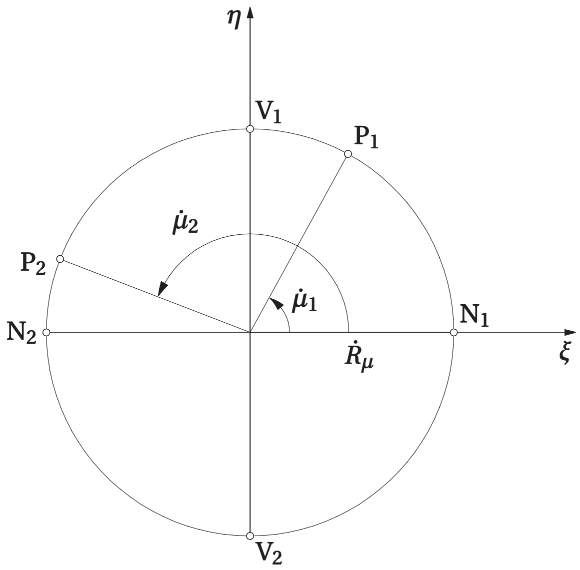

Figure 5. Equidistant circumference of the elliptical section of Figure 3

Table 1. Formulas to calculate R_μ and μ based on the third flattening n and the parametric latitude β (Orihuela, 2013)

$R_\mu = \frac{a}{1+n}\left(1 + \frac{1}{4}n^2 + \frac{1}{64}n^4 + \frac{1}{256}n^6\right).$	
$\mu = \beta + \sum_{j=1}^{j=6} D_j \sin\left(2j\beta\right),$	$\beta = \mu + \sum_{j=1}^{j=6} D'_j \sin\left(2j\mu\right),$
$D_1 = -\frac{1}{2}n + \frac{3}{16}n^3 - \frac{1}{32}n^5,$ $D_2 = -\frac{1}{16}n^2 + \frac{1}{32}n^4 - \frac{9}{2048}n^6,$ $D_3 = -\frac{1}{48}n^3 + \frac{3}{256}n^5,$ $D_4 = -\frac{5}{512}n^4 + \frac{3}{512}n^6,$ $D_5 = -\frac{7}{1280}n^5,$ $D_6 = -\frac{7}{2048}n^6.$	$D'_1 = \frac{1}{2}n - \frac{9}{32}n^3 + \frac{205}{1536}n^5,$ $D'_2 = \frac{5}{16}n^2 - \frac{37}{96}n^4 + \frac{1335}{4096}n^6,$ $D'_3 = \frac{29}{96}n^3 - \frac{75}{128}n^5,$ $D'_4 = \frac{539}{1536}n^4 - \frac{2391}{2560}n^6,$ $D'_5 = \frac{3467}{7680}n^5,$ $D'_6 = \frac{38081}{61440}n^6.$

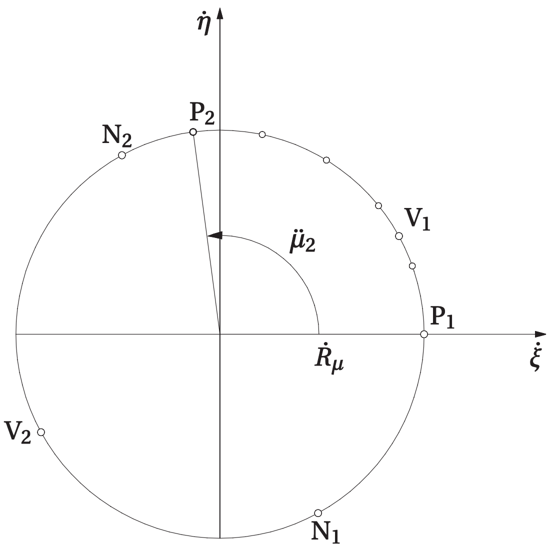

Figure 6. Rotated equidistant circumference that represents the elliptical section of Figure 3. Between P_1 and P_2, four waypoints are marked at equidistant angles

The points of the nodes and vertices of the section path correspond to the quadrants of the circle (Figure 5).

Calculating the angular difference according to the relative position of the points on the circumference may require the use of conditional sentences. To avoid this and so that the difference is positive and below 2π, it is convenient to rotate the coordinate system according to the angle of the first point,

$$\dot{\mathrm{R}} = \begin{bmatrix} \cos\dot{\mu}_1 & \sin\dot{\mu}_1 \\ -\sin\dot{\mu}_1 & \cos\dot{\mu}_1 \end{bmatrix},$$

applying this matrix, the following coordinate system is obtained (see Figure 6),

$$\begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} = \dot{\mathbf{R}} \begin{bmatrix} \dot{R}_\mu \cos\dot{\mu} \\ \dot{R}_\mu \sin\dot{\mu} \end{bmatrix}, \tag{21}$$

$$\ddot{\mu} = \arg\left(\dot{\xi}, \dot{\eta}\right). \tag{22}$$

The distance between any two points on the elliptical section is

$$s_{12} = \dot{R}_\mu\left(\ddot{\mu}_2 - \ddot{\mu}_1\right). \tag{23}$$

1.7. From the equidistant circumference to geodetic coordinates

The coordinate reversion depends solely on the formulas in subsections 2.5 and 2.6.

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \dot{\mathbf{R}}^{\mathrm{T}} \begin{bmatrix} \dot{R}_\mu \cos\ddot{\mu} \\ \dot{R}_\mu \sin\ddot{\mu} \end{bmatrix}; \tag{24}$$

$$\dot{\mu} = \arg\left(\xi, \eta\right); \tag{25}$$

$$\dot{\beta} = \dot{\mu} + \sum_{j=1}^{j=6} D_j^{'} \sin\left(2j\dot{\mu}\right), \tag{26}$$

where coefficients $D_j^{'}$ are extracted from Table 1.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \dot{a}\cos\dot{\beta} \\ \dot{b}\sin\dot{\beta} \end{bmatrix}; \tag{27}$$

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \mathbf{R}_Z^{\mathrm{T}} \mathbf{R}_X^{\mathrm{T}} \begin{bmatrix} \dot{x} \\ \dot{y} \\ 0 \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}, \tag{28}$$

where $\left[x_0, y_0, z_0\right]^{\mathrm{T}}$ are the coordinates of the center of the ellipse .

$$\begin{aligned} \phi &= \arg\left(r\left(1-e^2\right), Z\right); \\ \lambda &= \arg\left(X, Y\right), \end{aligned} \tag{29}$$

where $r = \left\| X, Y \right\|$.

2. Inverse problem

2.1. Algorithm of the inverse problem

For any elliptical section:

1. Calculate the ECEF coordinates of P_1 and P_2, formulas (1), (2) and (3).
2. Calculate the ECEF coordinates of P_0 which depend on the type of elliptical section, formula (4).
3. Calculate the perpendicular vector to the plane of the elliptical section, $\hat{\mathrm{n}}$, through $\hat{\mathrm{u}}$ and $\hat{\mathrm{v}}$, formulas (5), (6) and (7).
4. Determine the equation of the plane with normalized parameters, formulas (8) and (9).
5. Calculate the coordinates of the center of the elliptical section, $\mathbf{x}_0$ and the parameters of the ellipse, $\dot{a}$, $\dot{b}$ and $\dot{n}$, formulas (15), (16) and (17).
6. Calculate α_{12} and α_2 using the ENU components of P_1 and P_2, formulas (10), (11) and (12).
7. Change the orientation of the ECEF system to a plane coordinate system, calculate $\left(\dot{x}_1, \dot{y}_1\right)$, $\left(\dot{x}_2, \dot{y}_2\right)$, $\dot{\beta}_1$, $\dot{\beta}_2$, $\dot{\mu}_1$, $\dot{\mu}_2$ and $\dot{R}_\mu$, formulas (18), (19), (20) and Table 1.
8. Rotate the plane coordinate system according to $\dot{\mu}_1$; calculate $\left(\dot{\xi}_1, \dot{\eta}_1\right)$, $\left(\dot{\xi}_2, \dot{\eta}_2\right)$, $\ddot{\mu}_1$ and $\ddot{\mu}_2$, formulas (21), (22).
9. Calculate the length of the elliptical section, $s_{12} = \dot{R}_\mu\left(\ddot{\mu}_2 - \ddot{\mu}_1\right)$.

2.2. Numerical examples

The inverse algorithm is applied to the cases in the Table 2; the results are shown in Table 3 including, for comparison, the values corresponding to the geodetic line according to Karney's algorithm (2013). The numerical results presented in Table 3 are consistent in the order of millimeters with those shown in Gilbertson (2012, p. 3), although the mean normal section is defined differently; the ellipsoid used by Gilbertson (2012, p. 3) is not indicated.

Table 2. Test cases: 1 to 7 are extracted from Gilbertson (2012, p. 3), 8 is the example of Figure 2 and case 9 is an example with points near the antipodes, extracted from Karney (2013, p. 51)

Case	ϕ_1	λ_1	ϕ_2	λ_2
1	37.331931575000	0	26.128566516700	41.476529802800
2	35.269791283300	0	67.370771216667	137.791198430600
3	1.000000000000	0	–0.998286322222	179.296674991700
4	1.000000000000	0	1.020885977778	179.771622900000
5	41.696077777800	0	41.696166666667	0.000155555600
6	30.000000000000	0	37.892351622222	116.321302341700
7	37.000000000000	0	28.260193152778	–2.627646994400
8	60.000000000000	0	30.000000000000	80.000000000000
9	–30.000000000000	0	29.900000000000	179.800000000000

Depending on the desired accuracy, the distances of the sections and the geodetic line are very similar; however, the path is very different when the points are close to the antipodes. This case, taken from Karney (2013, p. 51), is illustrated in Figure 7: here, the paths between the different sections have the most notable differences.

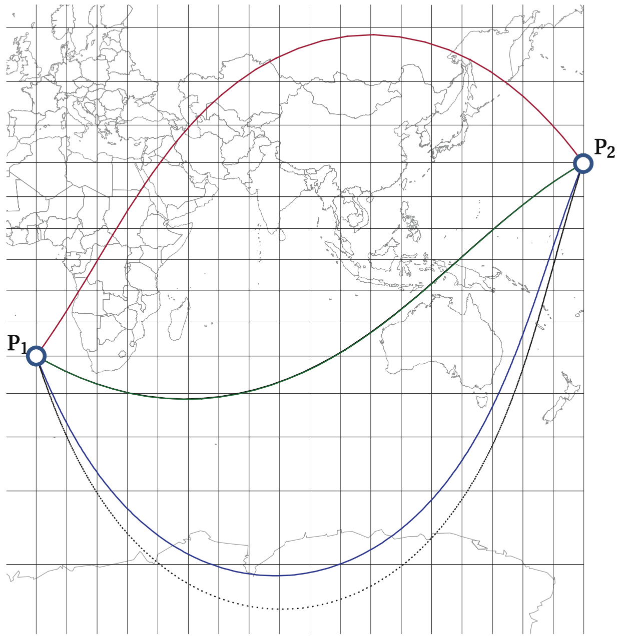

Figure 7. Illustration of case 9, Table 3, in a Mercator chart (GRS80 ellipsoid). From South to North: geodesic line (dotted line), first normal section (in blue), mean (in green) and central section (in black; both paths are confused in the chart) and second normal section (in red)

3. Direct problem

Because we need to know Z_0 at the time of posing the direct problem, it is only applicable to the central section and the first normal section.

3.1. Algorithm of the direct problem

1. Calculate the ECEF coordinates of P_1, formulas (1) and (3).
2. Calculate the ECEF coordinates of P_0, which depend on the central elliptical section or first normal section, formula (4).
3. Calculate the ENU components of P_1, formula (10).
4. Calculate the vector tangent to the elliptical section, $\hat{t}$, formula (13).
5. Calculate the vector perpendicular to the plane of the elliptical section, $\hat{n}$, through $\hat{u}$ and $\hat{t}$, formula (5), $\vec{n} = \hat{u} \times \hat{t}$ and $\hat{n} = \vec{n} / \|\vec{n}\|$.
6. Determine the equation of the plane with normalized parameters, formulas (8) and (9).
7. Calculate the coordinates of the center of the elliptical section, $\mathbf{x}_0$ and the parameters of the ellipse, $\dot{a}$, $\dot{b}$ and $\dot{n}$, formulas (15), (16) and (17).
8. Change the orientation of the ECEF system to a plane coordinate system, calculate $(\dot{x}_1, \dot{y}_1)$, $\dot{\beta}_1$, $\dot{\mu}_1$ and $\dot{R}_\mu$, formulas (18), (19), (20) and Table 1.
9. Rotate the plane coordinate system according to $\dot{\mu}_1$, calculate $(\dot{\xi}_1, \dot{\eta}_1)$ and $\ddot{\mu}_1$, formulas (21), (22).
10. Calculate $\ddot{\mu}_2$ according to the length of the elliptical section, $\ddot{\mu}_2 = s_{12} / \dot{R}_\mu + \ddot{\mu}_1$.
11. Calculate (ξ_2, η_2), $\dot{\mu}_2$ and $\dot{\beta}_2$, formulas (24), (25) and (26).
12. Calculate $(\dot{x}_2, \dot{y}_2)$ and ECEF coordinates of P_2, formulas (27) and (28).
13. Calculate the geodetic coordinates ϕ_2 and λ_2, formula (29).
14. Calculate α_2 using the ENU components of P_2, formulas (10), (11) and (12).

3.2. Consistency tests with the inverse algorithm

In Table 4, the application of the inverse algorithm is contrasted with the data in Table 2. The values are in decimal degrees. The differences converted to radians are of the same order as the truncation errors of double type

Table 3. Inverse algorithm applied to Table 2, a) central section, b) first section, c) second section, d) mean section, e) geodesic (Karney, 2017). The GRS80 ellipsoid is used

Case	α_{12}	α_2	s_{12}
1a	95.524334992059	118.048043737514	4085798.3231
1b	95.463168472388	118.109245795319	4085797.7189
1c	95.479913389005	118.092491115239	4085797.7343
1d	95.471538220679	118.100871163620	4085797.7125
1e	95.466906500992	118.100037749146	4085797.7105
2a	15.791303432383	144.869713292801	8084460.1567
2b	15.755076626803	144.905877052188	8084459.1586
2c	15.733449357830	144.927466667530	8084459.0171
2d	15.744248501486	144.916686326888	8084459.0453
2e	15.739863599959	144.927624307952	8084459.0129
3a	89.867403020768	90.144779253272	19959215.8296
3b	88.776954141822	91.235228165699	19959214.7524
3c	90.956084099565	89.056098200356	19959220.9103
3d	89.866468618648	90.145713655395	19959215.8270
3e	89.025504102167	90.976239608158	19959214.6263
4a	6.490942063115	173.509016585774	19779458.5035
4b	6.448311764271	173.551646882784	19779453.1247
4c	6.447427425187	173.552531221830	19779453.0147
4d	6.447869564525	173.552089082510	19779453.0697
4e	5.004745034151	174.995222922341	19779362.8384
5a	52.677183439521	52.677286524089	16.2833
5b	52.677183245480	52.677286717925	16.2833
5c	52.677183246422	52.677286718868	16.2833
5d	52.677183244462	52.677286716908	16.2833
5e	52.677183243912	52.677286716358	16.2833
6a	45.164346913679	128.964686405634	10002080.2789
6b	45.046799267671	129.082184042614	10002069.0165
6c	45.020017156451	129.108954727944	10002068.0591
6d	45.033402521686	129.095575074700	10002068.4630
6e	45.000084482765	129.136526168971	10002067.6835
7a	–165.003783117715	–166.418217417802	999975.5090
7b	–165.000014612863	–166.421987666777	999975.5084
7c	–165.000818247941	–-166.421183659669	999975.5084
7d	–165.000416336835	–166.421585756827	999975.5084
7e	–165.000275690648	–166.421458799242	999975.5084
8a	82.019793359039	144.992678079506	6623533.0390
8b	81.925502050507	145.087127664755	6623530.9089
8c	81.965324405885	145.047238382836	6623531.1326
8d	81.945391051954	145.067205222509	6623530.8971
8e	81.931524668844	145.068561860824	6623530.8637
9a	119.812789089792	60.087865277484	20000645.1302
9b	158.110377040057	21.789943762226	19989954.9854
9c	36.661670419061	143.238521275916	20010302.0144
9d	119.938021390319	59.962632345348	20000597.4530
9e	161.890524809390	18.090737172758	19989832.8275

numbers. Table 4 does not show the accuracy of the algorithms but the consistency of the inverse-direct operations with the original data.

Table 4. Consistency of the inverse and direct problem. Values α_{12}, α_2 and s_{12} are calculated with the inverse algorithm; α'_2, ϕ'_2 and λ'_2 are calculated with the direct algorithm. The cases are taken from Table 2. The difference is in decimal degrees

Case	$(\alpha'_2-\alpha_2)\times10^{15}$	$(\phi'_2-\phi_2)\times10^{15}$	$(\lambda'_2-\lambda_2)\times10^{15}$
1a	–25	–6	–32
1b	0	–6	6
2a	–51	38	–25
2b	–25	25	0
3a	0	0	0
3b	0	0	25
4a	–51	–9	0
4b	0	–22	0
5a	–6	–13	–7
5b	–6	–13	–20
6a	–25	–6	–25
6b	–25	38	–25
7a	0	–10	5
7b	0	3	5
8a	–25	6	13
8b	0	32	–13
9a	0	0	0
9b	–13	0	0

4. Waypoints

Building waypoints is a combination of inverse and direct problems. The number of points includes the final and initial points. We drew Figures 2, 4 and 7 with the help of the following algorithm:

4.1. Algorithm to build waypoints

1. Complete steps 1 through 9 of the inverse algorithm, (section "Inverse problem").
2. Given a certain number of points k, calculate the equidistant angular increment of the points, $\delta\ddot{\mu}=\left(s_{12}/\dot{R}_{\mu}\right)/(k-1)$.
3. Find the rectified angle for each point, $\ddot{\mu}_i=\ddot{\mu}_{i-1}+\delta\ddot{\mu}$.
4. For point $\ddot{\mu}_i$, calculate $\left(\xi_i,\eta_i\right)$, $\dot{\mu}_i$ and β_i, formulas (24), (25) and (26).
5. Calculate $\left(\dot{x}_i,\dot{y}_i\right)$ and the ECEF coordinates of P_i, formulas (27) and (28).
6. Calculate geodetic coordinates ϕ_i and λ_i, formula (29).
7. Calculate α_i with the ENU components of P_i, formulas (10), (11) and (12).
8. Repeat steps 3 to 7 until the number of points is completed.

An example is given in Table 5.

Table 5. Example of the path of the first normal section of case 8 (Table 2), GRS80 ellipsoid

Point	ϕ	λ	α
1	60.000000000000	0.000000000000	81.925502050507
2	60.269175390318	13.254222266165	93.432237828252
3	59.226197861025	26.201862952712	104.629189863010
4	56.995400315997	37.974562685100	114.637125249651
5	53.796126315162	48.157293005483	123.029798100222
6	49.860098676240	56.751503061822	129.796812992010
7	45.383235868353	63.974206036611	135.142396468560
8	40.514180802185	70.097147265708	139.324201305814
9	35.360660410871	75.370376432333	142.576444522586
10	30.000000000000	80.000000000000	145.087127664755

Conclusions

There are some advantages of elliptical sections with respect to geodesics: they are easily rectifiable plane curves and the path is unique when one or more full turns of the ellipsoid are made. The central section is unique for the two points and the direct and inverse problem can be solved; while, there are three types of normal sections to connect two points.The central section cannot replace the geodesic path when the points are close to the antipodes, although, proportionally the difference of the distances traveled is relatively small. Being a plane curve, the elliptical section has a solution to the inverse and direct problem, and waypoints without resorting to iterations, with the exception of the direct problem of the second and meannormal sections. The algorithms that solve this problem were described, which, unlike other approaches, make use of the equidistant circumference, reducing the length problem to the sum and difference of angles. The inverse and direct algorithms are consistent in the order of truncation error of double-type numbers.

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APPENDIX

It is not possible to calculate, without iterations, the direct problem of the second and meannormal sections. This is because we need to know the value of Z_0 . An iterative algorithm to solve these problems can be the following:

1. Solve the direct problem for the central section with ϕ_1 , λ_1 y α_{12} (section "Direct problem").
2. With the obtained value of ϕ_2' , calculate the ECEF coordinates of point P_0 , which depend on the second or mean normal section, formula (4).
3. With the new value of P_0 , solve the direct problem again until obtaining, ϕ_2 , λ_2 and α_2 . The value of ϕ_2 will be used to recalculate P_0 formula , $\phi_2' = \phi_2$.
4. Repeat steps 2 and 3 until $\left|\phi_2' - \phi_2\right| < 1\times10^{-15}$.

Table 6 shows the consistency of the inverse and direct problem of the second and mean normal sections, according to the examples in Table 2.

Table 6. Consistency of the inverse and direct problem. Values α_{12} , α_2 and s_{12} are calculated with the inverse algorithm; α_2' , ϕ_2' and λ_2' are calculated with the direct algorithm. The cases are taken from Table 2. The number of iterations is added

Case	$(\alpha_2' - \alpha_2)\times10^{15}$	$(\phi_2' - \phi_2)\times10^{15}$	$(\lambda_2' - \lambda_2)\times10^{15}$	Iter.
1c	–25	6	–32	5
1d	0	6	0	5
2c	–25	0	–25	5
2d	–25	25	–25	4
3c	0	0	–25	8
3d	0	0	–51	5
4c	0	4	0	4
4d	0	4	0	4
5c	–13	–13	–20	2
5d	–6	0	–7	2
6c	0	13	–25	6
6d	0	–19	25	6
7c	0	3	5	3
7d	0	54	5	3
8c	–25	32	–25	5
8d	–25	–6	0	5
9c	0	25	0	7
9d	25	25	51	6